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Pseudospin Symmetry in Heavy Deformed Nuclei.

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ABSTRACT

The origin and validity of pseudospin symmetry, which is characteristic for heavy atomic nuclei ($A \geq 100$), are discussed. The transformation to the pseudospin representation for physically significant operators is performed by means of specially designed analytical techniques.

The many-particle helicity operator is found to accomplish the transformation to the pseudospin basis in the scope of realistic nuclear models. Estimates based on both nonrelativistic mean-field and relativistic Dirac-Brueckner results show that in the helicity-transformed space nucleons move in a finite-depth, strongly nonlocal, potential with an effectively reduced spin-orbit strength. Since the chirality operation is the limiting case of the helicity operation for massless hadrons, the difference between the standard and helicity-transformed representation disappears in the high-energy chiral limit.

A procedure for applications of the pseudospin transformation within the framework of the spherical oscillator shell model is developed. It is valid for operators expressed in terms of single-particle variables and is based on permutation rules for special rotational invariants. The procedure is applied to a number of physical operators including several rotational scalars, the spin and orbital momenta, and the quadrupole moment. An algorithm for generating an approximation to the pseudospin transformation, which gives a simple and accurate expression for dominant parts of required transforms, is also given. The algebras associated with pseudospin transformations are considered. The analytical results thus obtained constitute the necessary input data for the nuclear structure calculations within the pseudo SU(3) algebraic collective model and its symplectic extension.

An operator, responsible for the transformation to the pseudospin representation within the oscillator shell model at an arbitrary triaxial deformation, is constructed as a special projection of the momentum helicity operator. Since the exact transformation of relevant operators cannot be performed in a closed analytical form, a procedure of approximate transformation is formulated as a generalization of the similar procedure for the spherical shell model. In the spherical and asymptotic prolate limits the transforms thus derived reduce to the familiar exact results. The approximate transform of a modified Nilsson Hamiltonian is found to be almost indistinguishable from the "pseudo" Hamiltonian with the strongly reduced spin-orbit strength.

CHAPTER 1

INTRODUCTION

A pseudospin (more precisely, pseudo space-spin) concept is a promising advance relating to the development of a shell-model theory for heavy ($A \geq 100$) nuclei. It was introduced independently by Hecht and Adler [34], and by Arima, Harvey and Shimizu [1] in 1969. The concept considers assigning of new spin and orbital momentum labels (so called pseudo labels) to individual nucleons in accordance with an observed near degeneracy of certain normal parity eigenstates (pseudospin doublets) of the spherical nuclear mean field. The spin-orbit interaction in the pseudo representation becomes rather weak (by almost an order of magnitude less) compared to the normal physical representation and this creates numerous conceptual and calculational advantages, especially in the scope of the many-particle, shell-model approaches to the low-energy nuclear structure.

Consider an example of the *sdg* oscillator shell which corresponds to the number of oscillator quanta $n = 5$. Because of the spin-orbit splitting, the set of single-particle energy levels consists of six degenerate subsets with the following values of the orbital and total angular momenta: $p_{1/2}$, $p_{3/2}$, $f_{5/2}$, $f_{7/2}$, $h_{9/2}$ and $h_{11/2}$. All of these levels but the $h_{11/2}$ belong to the same realistic nuclear shell with 82 and 126 nucleons as the lower and upper mass boundaries, respectively. Because of the strong spin-orbit interaction, the level with the maximal angular momentum defects into the lower shell, with $50 < N, Z \leq 82$. Among the levels remaining within the realistic shell (so called normal parity levels) there are two pairs which are nearly degenerate, namely, $p_{3/2}$, $f_{5/2}$ and $f_{7/2}$, $h_{9/2}$. The labels of both pairs can

be described by a common formula

$$\{(\tilde{l}-1)_{\tilde{l}-1/2},\,(\tilde{l}+1)_{\tilde{l}+1/2}\}$$

with $\tilde{l}=2$ and $\tilde{l}=4$, respectively.

Now observe that if the orbital momenta within each of the pairs are formally switched to $\tilde{l}$, and the angular momenta are retained, then after the relabeling one has to deal with the pairs of pseudo levels $\tilde{d}_{3/2}$, $\tilde{d}_{5/2}$ and $\tilde{g}_{7/2}$, $\tilde{g}_{9/2}$. Since the levels within each of the pairs are weakly split, it is possible to interpret them as components of pseudospin doublets provided the pseudo spin-orbit interaction is weak. Thus, the pseudo spin-space concept refers to a division of the single-particle total angular momentum operator into *pseudo* ($\mathbf{j}=\tilde{\mathbf{l}}+\tilde{\mathbf{s}}$) rather than *normal* ($\mathbf{j}=\mathbf{l}+\mathbf{s}$) orbital and spin parts. The corresponding rule of relabeling the values of angular momentum is very simple:

$$\tilde{l}=2j-l,\;\tilde{s}=s=\frac{1}{2},\;\tilde{j}=j.$$

This construction leads to noteworthy consequences which are considered below.

First, the relabeling rule maps the $p_{1/2}$ level onto $\tilde{s}_{1/2}$. Therefore, in the pseudospin representation, the normal parity levels of the *sdg* shell are mapped onto the complete $\tilde{p}\tilde{f}$ shell. Generally speaking, the normal parity sector of the nth shell of the spherical oscillator is put into one-to-one correspondence with the full $(n-1)$st shell of the spherical pseudo oscillator. The defector levels are left out of the consideration because of the large energy gap — this assumption is reasonable for the low-lying part of the nuclear spectrum. The concept then offers an effective

description of the many-particle nucleus through another system with noticeably fewer number of particles — which is a tremendous numerical simplification.

Second, the near degeneracy of the single-nucleon levels and, consequently, the grounds for introducing the pseudospin representation, is not an artifact of the oscillator shell model. To the contrary, this near degeneracy was observed in more realistic mean-field approaches as well: *e.g.*, the Woods–Saxon calculations in most cases reproduce very closely the same sequence and location of the states [4, 44]. Thus, the pseudospin concept is model-independent and reflects some natural trend. Nevertheless, the transformations to the pseudo representation in the scope of the oscillator shell model and of more realistic models occur to be somewhat different — and this is one of the topics of the current project.

Third, the sphere of applicability of the concept extends far beyond the mean-field theories. All the many-particle approaches, using the shell-model basis in one or another form, can actually benefit from the pseudospin representation, at least while the collective nuclear dynamics is concerned. This is especially true for algebraic models that utilize the symmetries of physical systems. For instance, the Elliott SU(3) collective model, which made explicit use of the oscillator symmetries and produced a breakthrough in understanding and description of rotational spectra of light nuclei [22, 23, 24, 33], occurred to be applicable to heavy deformed nuclei as well. This happened when Ratnu Raju, Draayer and Hecht realized that the weakness of pseudo spin-orbit splitting restored the oscillator SU(3) symmetry within the shell model broken by the strong spin-orbit forces in the normal representation [42]. Since then the pseudo-SU(3) model, and its Sp(6,R) extension (see reviews [26, 43] and references therein), which looks after the monopole and quadrupole modes, has

developed into a powerful tool for microscopic studies of collective phenomena in strongly deformed nuclei [11, 19].

The pseudospin concept was also extended to incorporate axial deformations [18, 21, 44] which remove most of the degeneracy characteristic of the spherical mean field. In this case the pseudo labels are still valid for the projections of the spin and orbital momenta on the body-fixed symmetry axis, and the pseudospin doublets are easily distinguishable in the spectra of realistic single-particle Hamiltonians. The pseudospin symmetry, whose goodness is determined by the weakness of pseudo spin-orbit interaction, has been successfully applied to various nuclear phenomena including superdeformation [21], identical bands [20, 38] and double beta decay [14]. In the framework of the pseudo SU(3) and pseudo symplectic models, the many-particle, shell-model calculations were accomplished for the collective spectra and transitions in heavy deformed nuclei [10, 42, 46].

Despite the empirical evidence in favor of the pseudo space-spin concept and its numerous applications for calculating shapes, spectra and transitions in specific nuclei, the attitude of nuclear theorists to this concept up to the recent period remained predominantly one-sided. Namely, the pseudospin symmetry, discovered from the mean-field calculations, for more than two decades was viewed mainly as a tool for constructing an effective coupling scheme. An important exception was Ref. [9] where a principal issue, of the operator responsible for the transformation to the pseudo representation, was raised for the first time. Recently, a number of articles appeared discussing an explicit form of this operator in terms of single-nucleon variables [3, 13, 17, 15] and its possible connection to the symmetries of nuclear interactions [2, 6]. In contrast to the rest of publications on pseudospin, those papers identified some questions of a more fundamental nature and looked towards

possible answers. A brief review of what was achieved will bring the objectives of the current project into better focus.

In 1982 Bohr, Hamamoto and Mottelson noticed that under the action of the coordinate helicity operator

$$U_r = i\frac{\mathbf{r}\cdot\boldsymbol{\sigma}}{r},$$

the single-particle spin-orbit operator, $\mathbf{l}\cdot\boldsymbol{\sigma}$ and the squared orbital momentum $\mathbf{l}^2$ transform in the following manner:

$$\begin{aligned} U_r\,\mathbf{l}\cdot\boldsymbol{\sigma}\,U_r^+ &= -\mathbf{l}\cdot\boldsymbol{\sigma} - 2, \\ U_r\,\mathbf{l}^2\,U_r^+ &= \mathbf{l}^2 + 2\mathbf{l}\cdot\boldsymbol{\sigma} + 2. \end{aligned}$$

(In these equations, $\boldsymbol{\sigma} = 2\mathbf{s}$ is the standard notation for the vector of Pauli matrices). Observe that in the basis of eigenstates of both the total angular and orbital momentum, these transformation rules exactly correspond to the rules of pseudospin relabeling. Moreover, if a wavefunction is constructed as a product of a radial function and the tensorial harmonic, relevant for the above case, the coordinate helicity transformation preserves the radial part but transforms the spin-angular one again in correspondence with the relabeling rules. There also exists a well-known phenomenological rule [28, 32, 40] that the single-particle level splitting within one shell of a spherical nucleus is given by a simple formula,

$$-k(\mathbf{l}\cdot\boldsymbol{\sigma} + \mu\mathbf{l}^2),$$

where the value of k depends on a region of the periodical system, and μ is a parameter close to 0.5 for heavy nuclei (in fact, it is about 0.4 for neutrons and 0.6

for protons). After the helicity transformation, this formula turns into

$$-k\left((2\mu-1)\mathbf{l}\cdot\boldsymbol{\sigma}+\mu\mathbf{l}^2-2(1-\mu)\right),$$

and the spin-orbit strength gets drastically reduced when the empirical values of μ are used. And, last but not least, the transformation is unitary and leaves the energy spectrum unchanged.

These observations allowed Bohr *et al.* to assume that the coordinate helicity transformation was responsible for the passage to the pseudospin representation. In his later analysis, Draayer [17] welcomed the consideration of the phenomenological formula but cast doubt on some other points of their result. First, there was a problem with applying the coordinate helicity transformation to a realistic mean-field Hamiltonian since it did not commute with the kinetic energy. Second, this transformation definitely did not coincide with the oscillator shell-model pseudospin transformation because it did not decrease the number of oscillator quanta by 1.

The explicit form of the pseudospin transformation for the spherical oscillator shell model was soon found by Castaños, Moshinsky and Quesne [13]. They substituted the coordinate vector $\mathbf{r}$ from the helicity operator by the boson annihilation operator $\mathbf{b}$ and made necessary corrections to keep the transformation unitary. However, the unitarity of this transformation happened to be restricted to the normal parity sector only; the subspace of defector states was projected out in accordance with the familiar constraint for the pseudospin representation. From an algebraic perspective, the transformation thus obtained is a supersymmetric operation constructed out of the rotational scalars which form the symplectic superalgebra osp(1|2) [3].

Two years later, Castaños, Velázquez, Hess, and Hirsch analyzed the structure of nearly degenerate levels in the single-nucleon spectra at very strong prolate deformations in terms of the cylindrical limit of the modified harmonic oscillator model (so called asymptotic Nilsson scheme) [15]. They came up with an explicit formula for the asymptotic transformation which was different from the spherical limit formula. A decomposition of the complete space of states into two subspaces — one where the transformation is unitary, and the other where the defector states are located — was also different. No explanation for the reason behind this difference was offered by the authors.

Another side of the pseudospin problem was considered by Bahri and Draayer [2]. While the previous authors were puzzled by the question *how* the pseudospin transformation appeared, these authors searched for the reason *why* it had that form. In fact, they estimated the value of the μ coefficient in the above mentioned phenomenological rule by using the coupling constants from the relativistic mean-field nuclear models and obtained an agreement with the empirical average $\mu = 0.5$ within 40%.

All these studies found some keys to understanding of pseudospin and, naturally, raised new questions. Some of these questions, conditionally divided into three groups, are presented below as a motivation for this project:

- What is the origin of pseudospin symmetry? Is it related to the symmetries of nucleon interactions? What transformation is responsible for the mapping of the normal physical representation onto the pseudo representation in the scope of realistic nuclear models? Is it the same as given by Ref. [13] or different? If different, what is its connection to the coordinate helicity transformation?

- How do the operators of physical importance, written in terms of the single-particle variables, change under the oscillator shell-model pseudospin transformation? Is there any simple analytical form, either exact or approximate, of the transformed many-particle operators that is convenient to use within the algebraical models like pseudo SU(3) and its extensions?

- What is the relation between the pseudospin representations (and transformations to those representations) for the oscillator shell model and more realistic models? Does this relation provide a key for explaining the deformation dependence of the oscillator pseudospin transformation? How does the pseudo spin-orbit strength depend on deformation?

The first group of questions, which deals with the origin of pseudospin, is considered in Chapter 2. The primary subject of this chapter is what stands beyond the conventional, *i.e.* confined to the framework of the oscillator shell model, understanding of the pseudospin symmetry. An explicit form is found for the transformation responsible for the passage to the pseudospin representation within the realistic nuclear models. Consequences of this corollary are analyzed by using some results from both nonrelativistic mean-field and relativistic Dirac–Brueckner approaches.

While the spherical oscillator pseudospin transformation is somewhat different from its microscopic precursor, it has an unconditional significance of its own because of the key role of the oscillator shell model in nuclear theory. This is the reason why the analytical techniques and specific results of the pseudospin transformation of physical operators, obtained in Chapter 3, can be of real help in various studies of the structure of heavy deformed nuclei.

The question of the deformation dependence of the oscillator pseudospin transformation is discussed in Chapter 4. The operator, responsible for this transfor-

mation, is constructed for an arbitrary deformation as a special projection of the microscopic operator derived in Chapter 1. Since the exact transformation of the physically interesting operators cannot be performed in a closed analytical form, a procedure of approximate transformation is developed. The transformation of a modified single-nucleon harmonic oscillator Hamiltonian is considered in detail. The results of the chapter comprise theoretical foundation for using the pseudospin transformation over the experimentally attainable domain of deformations.

The last chapter contains Summary and Conclusions.

CHAPTER 2

ORIGIN OF PSEUDOSPIN

Good pseudospin symmetry in heavy nuclei, while experimentally well corroborated and successfully used in numerous theoretical applications (see [19] for references), still lacks a sound microscopic explanation. A conventional level of understanding is based on the single-particle Hamiltonian of the oscillator shell model, namely, on the fact that deviations from the oscillator energy spectrum approximately follow a $2j(j+1)-l(l+1)$ dependence, which turns into $\tilde{l}(\tilde{l}+1)$ after the *normal→pseudo* relabeling (see Introduction). Relativistic nuclear mean-field estimates were presented [2] in support of such a dependence in the limit of large nucleon number. Also, a unitary operator was proposed [9] which acts on the spin and angular variables and accomplishes the *normal→pseudo* relabeling within a given shell; later this approach was revisited [13] in order to remove the one-shell restriction and resulted in the introduction of the operator which is unitary only within the normal–parity oscillator subspace.

This chapter is aimed at going beyond the scope of conventional explanations and based on the assumption that the pseudospin symmetry, which reveals itself on the single-particle (mean-field) level, has a microscopic origin related to the nature of internucleonic forces. The idea is to show that the main properties of the oscillator nuclear shell model pseudospin transformation are not so accidental, as it is still believed by some researchers, but rather attributable to a specific symmetry of the underlying microscopic interaction among nucleons. In other words, it is expected that there should exist a transformation, expressible in microscopic terms

(coordinates or momenta and spins of all nucleons) which acts on a many-body nuclear Hamiltonian in a manner similar to the way the pseudo-spin transformation acts on the Nilsson single-particle Hamiltonian of the oscillator shell model (for the sake of simplicity and consistency with the standard pseudo-spin concept, only the case of spherical nuclei is being considered). Namely,

- the transformation under discussion, like the pseudo-spin one, should commute with, and therefore preserve, the total angular momentum of each individual nucleon;

- the central part of a single-particle potential corresponding to the transformed many-particle Hamiltonian must not be significantly different form the initial (i.e. observed) nuclear potential which is known not to be affected strongly by the pseudo-transformation. The shape of this potential for heavier nuclei resembles a spherical well of a finite depth with a smoothed surface behavior. Thus, after the transformation, the radial part of the potential would at least retain its magnitude and flatness in the bulk;

- the spin-orbit part of the transformed single-particle potential should have essentially (about three to four times) reduced magnitude compared to the observed strength. Moreover, the proton and neutron components of the transformed potential would gain opposite signs — in contrast to the real system where both are attractive — thus causing a kind of an overall compensation for spin-orbit forces.

A unitary spin-angular microscopic transformation, different from those cited above [9, 13], is shown to fulfill key requirements for the pseudospin transformation when applied to the nucleus as a whole.

2.1 MICROSCOPIC PSEUDOSPIN TRANSFORMATION

To incorporate both single- and many-particle aspects of the pseudospin picture, a microscopic operator, responsible for the *normal→pseudo* transformation, should be of the form

$$U_{total} = \prod_{i=1}^{A} U(\mathbf{r}_i, \mathbf{p}_i, \boldsymbol{\sigma}_i), \tag{2.1}$$

where $\mathbf{r}_i$ stand for the position, $\mathbf{p}_i$ for the momentum, and $\boldsymbol{\sigma}_i$ for Pauli spin matrices of individual nucleons. The structure of $U(\mathbf{r}, \mathbf{p}, \boldsymbol{\sigma})$ is determined by the following general constraints:

a) $\tilde{\mathbf{l}}^2 = U\mathbf{l}^2U^{-1} = \mathbf{l}^2 + 2\mathbf{l}\cdot\boldsymbol{\sigma} + 2 = 2\mathbf{j}^2 - \mathbf{l}^2 + \frac{1}{2}$—this constraint sets the transformation rule [9];

b) $[U, \mathbf{j}] = 0$—rotational invariance;

c) $[U, \mathcal{P}] = [U, \mathcal{T}] = 0$—parity and time-reversal symmetry;

d) $UU^+ = U^+U = 1$—unitarity and conservation of observables;

e) $[U, \mathbf{p}] = 0$—translational invariance.

Once constraints a), b), c) and d) are applied along with the heuristic requirement of relative simplicity, only three choices for U remain:

$$U = (d_\theta\, d_\theta^+)^{-1/2} d_\theta, \quad d_\theta = (\cos\theta\, r_0\mathbf{p} + i\sin\theta\, \mathbf{r}/r_0)\cdot\boldsymbol{\sigma}, \tag{2.2}$$

where r_0 is a characteristic length. Due to the option of rescaling r_0, the value of θ can always be set equal to $\pm\frac{\pi}{4}, 0$ or $\frac{\pi}{2}$. The first of these choices yields the operator of Ref. [13], specifically designed for the oscillator shell model applications, or its Hermitian conjugate. However, this operator is unitary only within the subspace of normal parity states but undefined in the unique parity subspace.

When the global unitarity is required, two possibilities remain. The case of $\theta = \frac{\pi}{2}$ corresponds to the $U_r = i\boldsymbol{\sigma}\cdot\mathbf{r}/r$ operator (henceforth referred to as r-helicity), proposed in Ref. [9]. The $\theta = 0$ choice is the p-helicity, $U_p = \boldsymbol{\sigma}\cdot\mathbf{p}/p$. The latter operator is an only one *compatible with the constraint of translational invariance* and thus consistent with the realistic many-particle approach.

The momentum helicity and oscillator pseudospin transformation are closely related. In the next section the helicity transformation is analyzed on the level of basis functions, and the relation between the two transformations is considered in more detail. The results obtained are used afterwards as a part of the subsequent comparative discussion of the r- and p-helicity transformations.

2.2 HELICITY AND PSEUDOSPIN TRANSFORMATIONS

The many-particle helicity transformation can be written in a multiplicative form for the system of A nucleons:

$$U_{r,total} = \prod_{i=1}^{A} \frac{\mathbf{p}_i \cdot \boldsymbol{\sigma}_i}{p_i}, \tag{2.3}$$

where $\mathbf{p}_i$ stand for the momentum, p_i for its absolute value, and $\boldsymbol{\sigma}_i = 2\mathbf{s}_i$ for the Pauli spin matrices of the individual particles. This is the only microscopic unitary transformation which yields the normal→pseudo transformation rules for the single-particle spin and orbital momenta and simultaneously preserves the translational invariance of the nucleon system. The conventional pseudospin transformation is therefore a version of the latter adapted for the oscillator shell model. The following analysis displays the relation between the two transformations in more detail.

Consider the action of the helicity transformation on the shell-model wavefunctions in the spherical representation. The nature of the shell model reduces the

problem to the single-particle level, and the relevant wavefunctions are written in the form [8, 9],

$$\psi_{nljj_z}(\mathbf{r},\sigma) = i^l R_{nl}(r)(\mathbf{Y}_l(\mathbf{e}_\mathbf{r}) \otimes \chi)_{jj_z}, \tag{2.4}$$

where n is the number of quanta, l, j and j_z denote orbital momentum, total angular momentum and its projection, respectively, $\mathbf{Y}_l(\mathbf{e}_\mathbf{r})$ is a spherical harmonic (in the direction of $\mathbf{e}_\mathbf{r}$, the unit vector along $\mathbf{r}$), and χ is a Pauli spinor. As shown in Appendix A, under the helicity operation the function (2.4) transforms as follows:

$$U_r\psi_{nljj_z}(\mathbf{r},\sigma) = i^{\tilde{l}}\mathcal{R}_{\tilde{n}\tilde{l}q}(r)(\mathbf{Y}_{\tilde{l}}(\mathbf{e}_\mathbf{r}) \otimes \chi)_{jj_z}, \tag{2.5}$$

where

$$q = 2(l-j),\ \tilde{l} = l - q,\ \tilde{n} = n - q. \tag{2.6}$$

The analytical expressions for the radial functions $\mathcal{R}_{nl+}(r)$ and $\mathcal{R}_{nl-}(r)$ are given by Eqs. (A.13), (A.14), and the value of the n quantum number is chosen equal to the shell number of the basis state providing the maximal contribution into the spherical oscillator basis expansion.

The radial dependence of the transformed functions differs from the radial dependence of the related shell-model functions. A direct comparison of those functions in Fig. (2.1) for the case of $n = 5$, $l = 3$ demonstrates that the transformed functions decrease rather slowly in the nuclear surface region while in the bulk they behave practically like the closest oscillator function, either slightly compressed ($q = 1$) or dilated ($q = -1$) (see Eq. (A.19)). The relevant oscillator basis expansion coefficients for the same case are plotted in Fig. (2.2) which demonstrates that the dominant shell contribution is 80-85%, and the higher shells effect the radial func-

tion's tail only. Since the bulk behavior of more realistic single-particle functions (for instance, for the Woods-Saxon potential) is very similar to the behavior of the oscillator ones, the above statement remains valid for realistic mean-field models.

Therefore, the helicity transformation of the basis functions in the spherical representation is nearly model-independent and can be accomplished, at least approximately, via simple prescriptions. However, the strong nonlocality of the helicity operation makes an expression for the transformed mean field rather inconvenient for standard analyses [6]. From this standpoint the conventional pseudospin transformation can be understood as a practical compromise, which allows, albeit in the framework of the oscillator shell model only, for a straightforward analytical construction of the images of the single-particle Hamiltonian and the basis states along with the correct transformation rules for the spin and orbital momenta.

The passage from the helicity transformation to the pseudospin one requires two major steps. First, the transformed functions are replaced by the closest oscillator functions, *i.e.* the $\mathcal{R}_{\tilde{n}\tilde{l}q}(r)$ factors in the r.h.s. of Eq. (2.5) change to $R_{\tilde{n}\tilde{l}}(r)$. Within the bulk of the nucleus this operation is well approximated by a simple rescaling. The operation is unitary in the entire Hilbert space of the oscillator states and maps the states of the shell n onto the states which belong to both the $n-1$ and $n+1$ shells, depending on the q value (see Eq. (2.6)). At the second step, $\tilde{n}$ is redefined to be equal to $n-1$ regardless of q; however, the relabeling rule for the orbital momentum is still given by Eq. (2.6). Thus, the pseudospin transformation maps the normal parity states of a given shell onto the complete shell with one quantum less; but the defector states no longer have their images. For instance, the $p_{1/2}$, $p_{3/2}$, $p_{5/2}$, $p_{7/2}$ and $h_{9/2}$ orbitals of the $n=5$ shell map onto the $\tilde{s}_{1/2}$, $\tilde{d}_{3/2}$, $\tilde{d}_{5/2}$, $\tilde{g}_{7/2}$ and $\tilde{g}_{9/2}$ orbitals of the $\tilde{n}=4$ shell, respectively, while the $h_{11/2}$ orbital has no image.

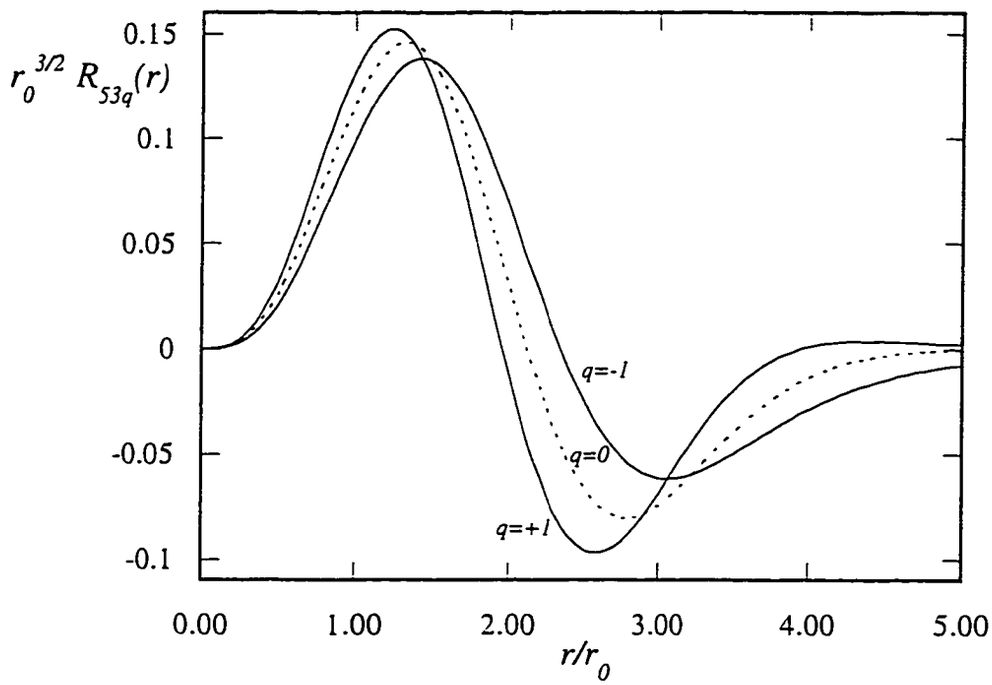

Figure 2.1: Helicity transformed oscillator radial functions for $n = 5$, $l = 3$. Calculated according to Eqs. (A.13) and (A.14). Dashed line displays the behavior of the closest oscillator function, $R_{53}(r)$, which is also calculable by setting $q = 0$.

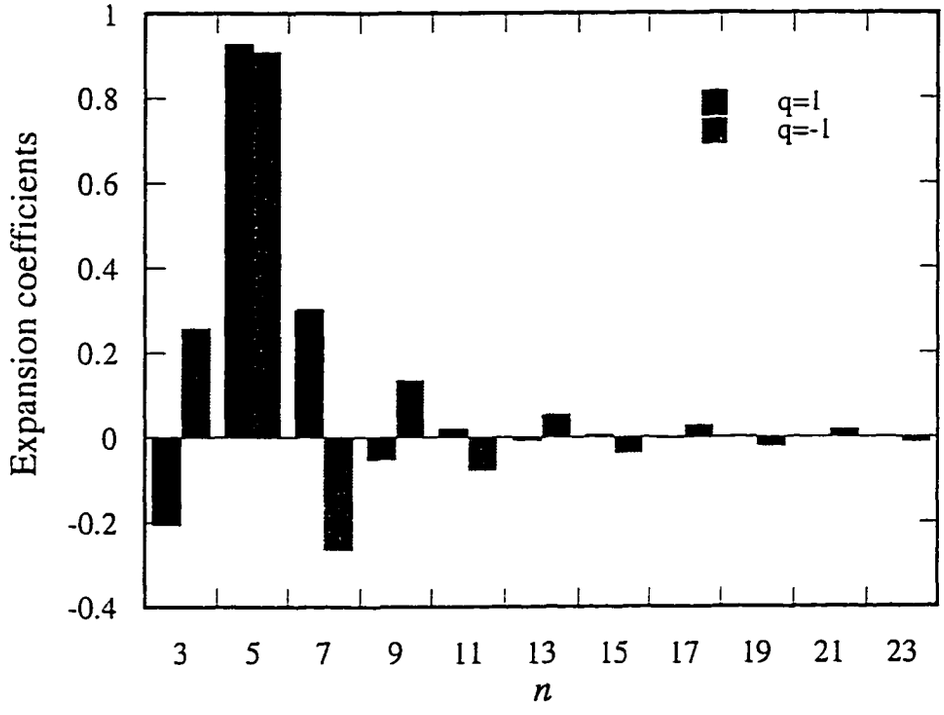

Figure 2.2: Expansion of helicity transformed functions with $n = 5$, $l = 3$ in terms of oscillator functions.

The pseudospin transformation is convenient for practical purposes in many aspects but retains the unitarity property only within the normal parity subspace. The latter subspace maps onto the entire space of states after the transformation, and in this sense the pseudospin scheme delivers only an effective description of the nuclear system for the low-energy region. Nevertheless, this description is found to be very reasonable in predicting energy spectra and shapes of heavy deformed nuclei within the framework of many-particle, shell-model approach [19]. Some refinement, based on microscopic treatment of the missing defector states, also brings the calculated transition rates into good agreement with experimental data [25]. As a whole, the pseudospin representation offers an approximate, although physically relevant, way of model description of the heavy nuclei. A special advantage of this description is the possibility of analytically finding the transforms of important physical operators.

2.3 Single-particle Hamiltonian and wavefunctions

If the transformation (2.1) is responsible for the pseudospin symmetry, then, in addition to the above constraints, it should nearly decouple the spin and orbit valence degrees of freedom in heavy nuclei. The applicability of the mean field approach allows for a reasonable direct check on this by considering the transformed single-particle Hamiltonians and wavefunctions. Corrections for the center-of-mass motion, relatively small for heavier nuclei, are not expected to worsen such an argument.

For simplicity, a spherically symmetric field is considered. In this case, the conventional form of the Hamiltonian is

$$H = \frac{\mathbf{p}^2}{2M} + V(r) + W(r)\mathbf{l}\cdot\boldsymbol{\sigma}, \tag{2.7}$$

and the wavefunctions are given in Eq. (2.4).

Under the r-helicity operation, Eqs. (2.7), (2.4) transform into

$$U_r H U_r^+ = \frac{\mathbf{p}^2}{2M} + V(r) - 2W(r) + \frac{\hbar^2}{Mr^2} + (\frac{\hbar^2}{Mr^2} - W(r))\mathbf{l}\cdot\boldsymbol{\sigma}, \tag{2.8}$$

$$U_r \psi_{nljj_z}(\mathbf{r},\boldsymbol{\sigma}) = i^{\bar{l}} R_{nl}(r)(\mathbf{Y}_{\bar{l}}(\mathbf{e_r}) \otimes \chi)_{jm}, \tag{2.9}$$

where $\bar{l}$ is determined by the known rule $l = j \pm \frac{1}{2} \rightarrow \bar{l} = j \mp \frac{1}{2}$. Note the striking difference between (2.4), the initial wavefunction, and (2.9), its r-helicity transform. Under the transformation the angular part of the wavefunction changes in accordance with the change of angular momentum while the radial function is not affected [9] and, consequently, has an abnormal behavior $\propto r^l$ instead of $r^{\bar{l}}$ at $r \rightarrow 0$. The reason for such a behavior is a noncommutability of the r-helicity operator with the kinetic energy, which generates $\frac{1}{r^2}$-corrections to both the central and spin-orbit potentials (see Eq. (2.8)). Moreover, since $W(r)$ is an attractive potential, the r-helicity operation can only increase the spin-orbit splitting in absolute value. This is an additional reason why the translationally non-invariant r-helicity transformation cannot be regarded as an appropriate microscopic pseudospin transformation.

To consider the p-helicity case, start from the coordinate representation for the unitary operator U_p:

$$U_p = -iK(\hat{l} - \Lambda - 1)^{-1} r(\boldsymbol{\sigma}\cdot\boldsymbol{\nabla}), \tag{2.10}$$

where $K = \Gamma(\frac{\hat{l}+\Lambda+2}{2})\Gamma(\frac{\hat{l}-\Lambda}{2})(\Gamma(\frac{\hat{l}+\Lambda+3}{2})\Gamma(\frac{\hat{l}-\Lambda-1}{2}))^{-1}$ is a unitary operator, $\hat{l} = \frac{1}{2}((1 + 4\mathbf{l}^2)^{-1/2} - 1)$ is the operator whose eigenvalues are quantum numbers of the orbital momentum, and $\Lambda = \mathbf{r}\cdot\boldsymbol{\nabla} = r\partial/\partial r$ is the infinitesimal shear operator. The unitarity of K follows from the conjugation rules $\Lambda^+ = -(\Lambda + 3)$, $\hat{l}^+ = \hat{l}$. Then Eq. (2.7) transforms into

$$\begin{aligned} U_p H U_p^+ &= \frac{\mathbf{p}^2}{2M} + K(V(r) - 2W(r) - (\hat{l}+1)v(r))K^+ \\ &\quad + K(v(r) - W(r))K^+ \mathbf{l}\cdot\boldsymbol{\sigma}, \end{aligned} \tag{2.11}$$

where $v(r) = (\hat{l}-\Lambda-1)^{-1}(rV'(r) - (\hat{l}+2)rW'(r))(\hat{l}+\Lambda+2)^{-1}$. In contrast with the previous case, the difference $(U_p H U_p^+ - H)$ is entirely originated by the potential energy transformation. Due to a complicated dependence on Λ in the r.h.s. of (2.11), the transformed potential energy operator is strongly *nonlocal*. Although (2.11) in its general form does not provide incontrovertible evidence for a reduction in the magnitude of the spin-orbit splitting, the latter is likely to happen at low $\tilde{l}$ within the nuclear surface region if an effective value of the Λ operator exceeds $\tilde{l} - 1$.

The transform of the wavefunction (2.4) is determined by Eq. (2.5). The radial functions, whose explicit form is given by Eqs. (A.13) and (A.14), show the following asymptotics

$$\bar{R}_{nj\tilde{l}}(r) \propto \begin{cases} r^{\tilde{l}}, & r \to 0, \\ r^{-(l+3)}, & r \to \infty. \end{cases} \tag{2.12}$$

The completeness of the oscillator basis validates Eq. (2.12) for more realistic cases including that of the present consideration. The standard $r^{\tilde{l}}$ dependence in the interior region follows because deep in the bulk of a heavy nucleus the transformed central potential is not expected to deviate significantly from the flat behavior of

$V(r)$. The $r^{-(l+3)}$ outer region asymptotics — it depends on the original orbital momentum l — means a more diffuse surface (with the nuclear density decreasing as r^{-6}).

Thus, the p-helicity transformation brings higher diffuseness and a strong non-locality of the potential in the surface region while performing the *normal→pseudo* relabeling of the angular momenta.

2.4 Estimates based on density-dependent OBEP

A relativistic extension of the Brueckner theory (see [30, 36] for references) provides parameter-free microscopic predictions for both infinite and finite nucleon systems. While it gives a good description of nuclear matter, the gross features of finite (especially lighter) nuclei are reproduced less well but nonetheless much better than in nonrelativistic approaches [37]. For this reason, the results of Dirac–Brueckner nuclear matter calculations are used below for reasonably estimating the p-helicity transformed two-body nuclear interaction, as well as the mean field, in heavier nuclei.

For a wide range of densities, including the saturation point, the nucleon-nucleon interaction in the infinite medium is perfectly approximated by a one-boson exchange potential (OBEP) with the boson parameters fitted to the Bonn model, and the density-dependent effective nucleon mass M^* calculated in a self-consistent manner [36]. To a very good approximation, the density-dependent self-consistent field has the same Lorentz structure as the free Dirac Hamiltonian. Consequently, a single-particle Hamiltonian *in the medium* commutes with the p-helicity, and the *helicity transformation does not affect the single-particle energies.*

However, the two-body interaction changes dramatically. In the representation of plane-wave Dirac spinor nucleon states, normalized to unity, which is the basis for Dirac–Brueckner calculations, the p-helicity operation is equivalent to $i\gamma^5 S$ when acting on the right (ket) states. Here $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is the product of Dirac matrices, and S is a formal operation for switching the sign of the effective mass: $Sf(M^*, \mathbf{p}) = f(-M^*, \mathbf{p})$. Since the γ-matrices simply switch sign under the chiral transformation γ^5, and the OBEP is bilinear in those matrices, the *helicity transformation of the OBEP is reduced to changing the sign of* M^* in the momentum representation. This is easily accomplished in the two-nucleon center-of-mass frame (simulating the center-of-mass frame of the nucleus) and produces strongly incident-energy dependent, *i.e.* nonlocal, interactions. The degree of nonlocality of the transformed OBEP is far beyond the level one deals with in a real world: since the helicity operator depends directly upon the the angles between spins and momenta, the transformed OBEP incorporates a strong dependence on the scattering angle even at very low relative momenta. Because only a rough estimate for the potentials is sought, here these potentials are converted into local approximations by averaging over allowed values of the relative momentum q with an appropriate distribution of q at a fixed momentum transfer k. The localized helicity-transformed OBEP in the momentum space converges rapidly in the shortwave region ($k > 2k_F$) to the initial potential, averaged with the same distribution. The values of the localized central part of the potential also coincide at $k = 0$ before and after the helicity transformation in accordance with the helicity-invariance of the single-nucleon energy in the medium.

The localized estimates for transformed single-particle potentials in the coordinate space, as given in Figs. (2.3) and (2.4), are calculated in the first order pertur-

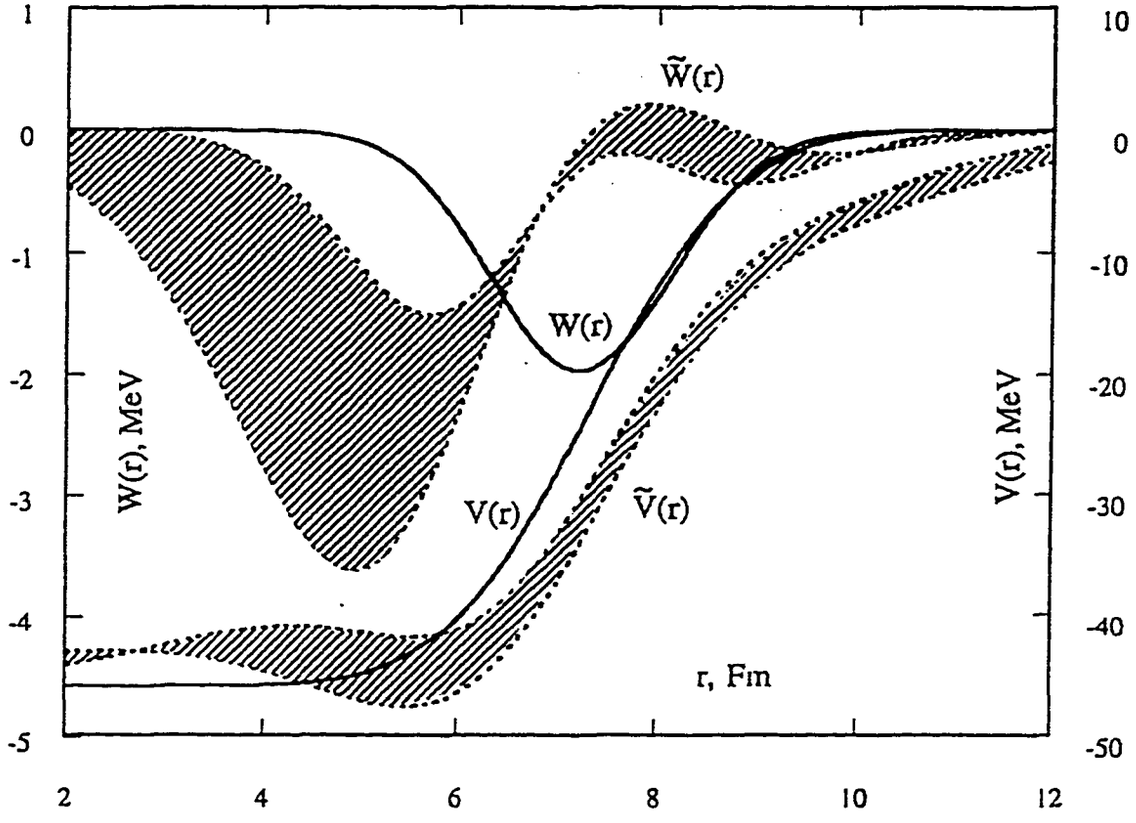

Figure 2.3: Localized estimates for the neutron central and spin-orbit potentials in ^{208}Pb before and after the helicity transformation (continuous lines and shaded areas, respectively). The two curves that define the borders of the shaded areas were determined by using different reasonable approximations for the relative momentum distribution in a finite nucleus ($5 \leq \eta \leq 9$ for the central potential, $1 \leq \eta \leq 3$ for the spin-orbit potential — see the explanation in the text.)

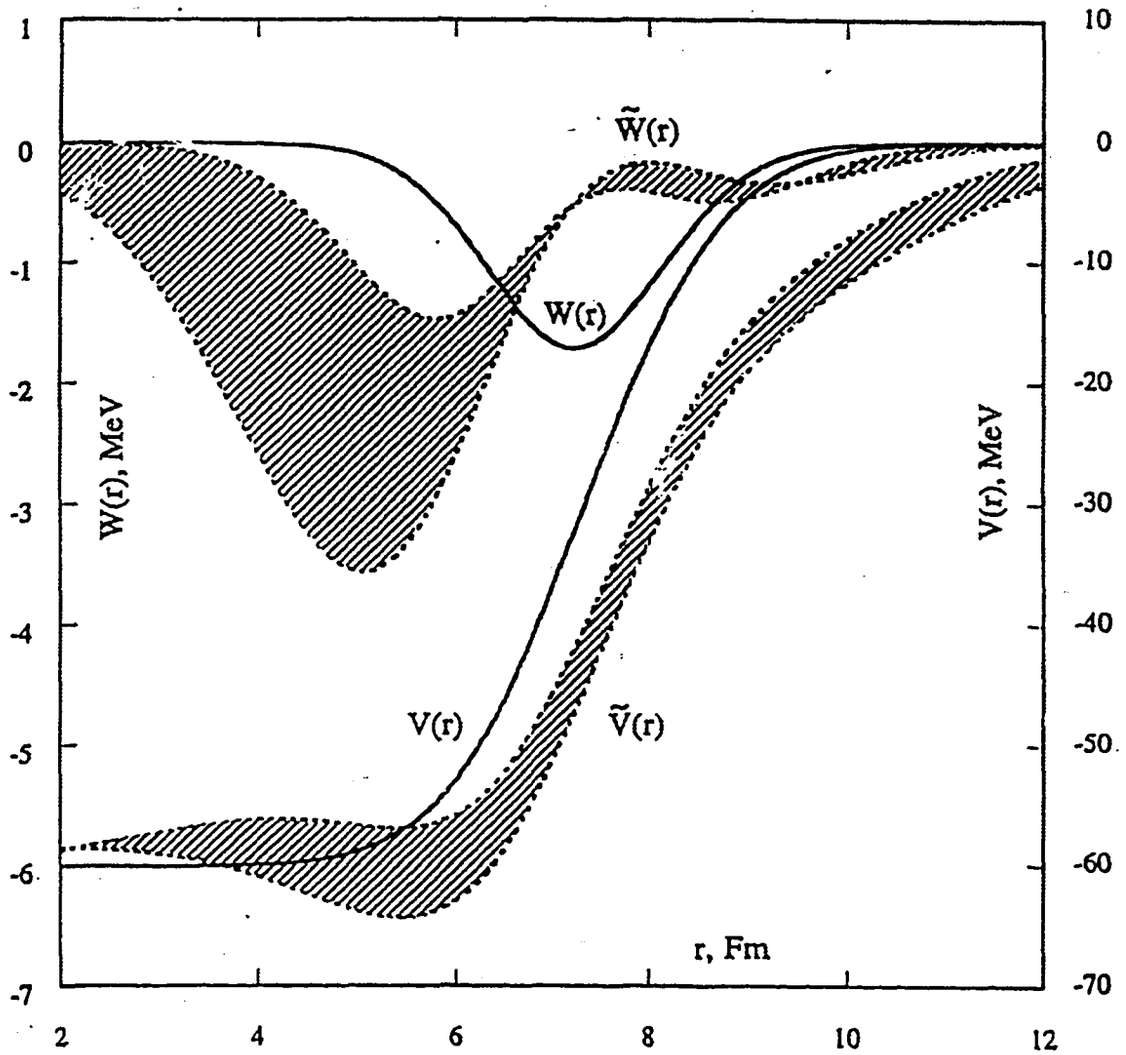

Figure 2.4: Localized estimates for the proton central and spin-orbit potentials in ^{208}Pb before and after the helicity transformation (continuous lines and shaded areas, respectively).

bation theory with respect to $\delta V(k)$, the localized difference between transformed and initial OBEP. Unperturbed potentials are taken in the standard Woods-Saxon parametrization [16] with the following adjustment for the radial dependence

$$(1+\exp(\frac{r-R}{a}))^{-1} \longrightarrow \frac{1}{2}(1-\mathrm{erf}(\sqrt{\frac{3}{2}}\frac{R-r}{\pi a})), \tag{2.13}$$

which allows for a simple analytic Fourier transform along with a quantitative fit. The estimate is done analytically using a low momentum expansion of the Skyrme type for $\delta V(k)$ and zero-order nuclear density distribution of the same kind as (2.13) but with a lesser diffuseness [8]. Due to the strong nonlocality of transformed OBEP the analytic formulas for single-particle potentials are more complicated than in the conventional scheme with Skyrme forces [41]. Basic complications and approximations are the following: a) d- and f-waves of the relative motion make an impact of the same order of magnitude as s- and p-waves that are normally included; b) $\delta G(k)$, a difference between the G-matrix, localized in the transformed space, and the physical G-matrix, coincides with $\delta V(k)$ in the first order because of a small contribution to the ($k > 2k_F$) region (see previous paragraph); c) instead of choosing a specific distribution for the relative momentum q, the k-dependence of averaged q^{-2} function (which is mostly responsible for nonlocalities) is set equal to a model function $4(1-\exp(-\frac{\eta k^2}{4k_F^2}))/k^2$ which mimics an effect of various distributions in the $k \to 0$ region by using different values of η and has correct asymptotics at $k \gg k_F$ (the upper limit $\eta = 9$ corresponds to the Fermi gas distribution in infinite nuclear matter; the lowest possible value $\eta = 1$ is associated with a distribution localized at $q = k_F$ and makes sense for surface effects); d) the ratio of proton and neutron densities is fixed for all r, and Coulomb corrections are not considered due to the

roughness of the procedure; e) M^* and k_F are fixed at their saturation point values [36].

Although the single-nucleon potentials shown in the figures are rough local estimates for the strongly nonlocal fields, they display several features characteristic of the pseudospin symmetry. First, in accordance with Sec. 2.2, the transformation preserves the finite depth of the central potential and increases the surface diffuseness. Second, a minimum of the spin-orbit potential, which is located in the surface region in the normal representation, gets shifted deeper into the bulk as a result of the helicity transformation. And from this it follows that the magnitude of the spin-orbit potential in the region where the wavefunctions are localized and which is primarily responsible for the interaction strength, exhibits a dramatic decrease. Also note that the effective *pseudo* spin-orbit interaction of the neutrons is more repulsive than one of the protons — in consonance with experiment [9].

2.5 Concluding remarks

The microscopic origin of the pseudospin symmetry has been considered. The many-particle p-helicity operator has been found to be the only one to generate the proper relabeling of the spin and orbital momenta while satisfying all other general symmetry requirements. The wavefunctions transform in a physically reasonable manner under the p-helicity operation, and the single-particle spin-orbit interaction strength is effectively compensated as compared to the normal (not pseudo) picture.

The effect of the helicity operator on the realistic mean-field Hamiltonian and the realistic many-particle Hamiltonian with density-dependent one boson exchange interaction (based on the Bonn model) has been considered in order to get a feeling for how the pseudospin transformation works on the microscopic level. Mean-field

estimates show that the transformed single-particle potential, similarly to the original one in the coordinate space, has a finite depth but becomes strongly nonlocal, i.e. state-dependent. The single-particle spin-orbit interaction becomes highly nonlocal as well. The estimates using the one-boson exchange potential lead to similar consequences. The analysis of the helicity-transform of this potential demonstrates that some components (spin-spin and tensor interactions) are invariant while others (including central spin-independent and spin-orbit forces) change rather drastically. The latter components — they give primary contributions to the single-particle potentials — acquire a very strong dependence on the incident energy in the momentum space which is equivalent to the nonlocality in the coordinate space. It is also noteworthy that the spin-orbit potential does not decrease in amplitude as one might expect — but rather is rearranged in coordinate space so as to reduce the spin-orbit splitting of single-particle states, which is a space-average property of the interaction itself.

The approximate independence of the single-nucleon spectrum in an infinite medium on the helicity transformation and the consistency of the microscopic estimates for the single-particle nuclear potentials with the Dirac–Brueckner calculations indicates a connection of the pseudospin symmetry to the boson-exchange nature of nucleon-nucleon interactions.

Another intriguing feature of the helicity transformation is its close relation to the chirality transformation familiar from high-energy studies. Indeed, in the chiral limit of massless hadrons there is no difference between these two operations, and, therefore, the normal and helicity representations coincide. The difference comes with the mass because the helicity transformation keeps the kinetic energy invariant and changes the interparticle forces. However, for the infinite matter in

the energetic domain where the nucleons are well established, the difference between the two representations is still rather small since the average field necessarily has the Lorentz-covariant structure and thus commutes with the helicity operation. For finite nuclei the Hamiltonians before and after transformation remain isospectral — this is consequence of unitarity — but, as discussed, a significant difference occurs in the dynamics of corresponding degrees of freedom. This line of thought leads to a suggestion that the origin of the pseudospin may be connected to the chiral symmetry. Putting it the other way around, the goodness of pseudospin symmetry may be expected to increase with raising density (or energy per particle) in hadronic systems, and actually yield to chiral symmetry in the region of asymptotic freedom. Indeed, due to relatively small values of s- and d-quark masses, the results seem to suggest that the quark models in a broad energy domain might benefit from an introduction of the pseudospin basis.

Chapter 3

Pseudospin Transformation of Physical Operators

The transformation from the normal representation to the pseudo space-spin representation, usually abbreviated the *normal→pseudo* transformation or simply pseudospin transformation, is conventionally defined within the framework of the harmonic oscillator shell model. It can be viewed as simply a scheme for relabeling the single-nucleon components of the oscillator shell-model states associated with the normal parity subspace [17]. Although this interpretation is too restrictive to be directly applicable in realistic mean-field and many-particle descriptions of nuclei, which require instead the helicity transformation [6], it is of primary importance because of the key role the oscillator shell model plays in microscopic nuclear calculations.

An alternative interpretation for the normal→pseudo transformation in terms of the single-particle coordinates, momenta and spins variables has been suggested [13]. The relabeling of single-particle states, which up until recently was the exclusive tool for effecting the normal→pseudo transformation, is a working procedure that is well-suited to numerical calculation within a given model space. From the standpoint of operators, however, it only yields simple results for actions defined on single-particle basis states. For example, the relabeling procedure is very simple when applied to the pairing interaction which creates and annihilates pairs in time-reversed nucleon states [45]. However, the majority of physical operators, including the kinetic energy, internucleon potentials, electromagnetic transitions, and so on,

are most easily expressed in terms of the variables r, p and s. For these forms the relabeling can only be effected in terms of a second quantized representation which must be done numerically for every major shell by means of the symmetry-adapted tensorial expansion and this, in turn, complicates the interpretation of the result [42].

In this chapter, the analytical form of the normal→pseudo transformation is used in constructing a procedure for finding transforms of operators that have an explicit dependence on single-particle variables. This procedure constitutes a basis for developing an analytical formalism and carrying out calculations within any microscopic theory using the pseudo space-spin concept, especially the pseudo-SU(3) and pseudo-Sp(6,R) theories. The analytical results are valid for any oscillator shell. The transformation of several important operators, including the spin and quadrupole moment, is discussed in detail. The images that are obtained are compared with tensorial expansions derived using the relabeling algorithm. A heuristic technique is developed to deduce simple approximations to the normal→pseudo images which extract the dominant parts in a simple and accurate manner.

3.1 PROPERTIES OF PSEUDOSPIN TRANSFORMATION

For the system of A nucleons, the normal→pseudo transformation can be written in a multiplicative form [13],

$$U_{total} = \prod_{i=1}^{A} U(\mathbf{r}_i, \mathbf{p}_i, \boldsymbol{\sigma}_i), \tag{3.1}$$

where $\mathbf{r}_i$ stand for the position, $\mathbf{p}_i$ for the momentum, and $\boldsymbol{\sigma}_i = 2\mathbf{s}_i$ for the Pauli spin matrices of the individual particles. The corresponding single-particle operators

$U(\mathbf{r}, \mathbf{p}, \boldsymbol{\sigma})$ are representable as follows [13, 3]:

$$U(\mathbf{r}, \mathbf{p}, \boldsymbol{\sigma}) = d_0 \, (d_0^+ d_0)^{-1/2}, \tag{3.2}$$

$$d_0 = \mathbf{b} \cdot \boldsymbol{\sigma}, \; d_0^+ = \mathbf{b}^+ \cdot \boldsymbol{\sigma}, \tag{3.3}$$

where $\mathbf{b} = (\mathbf{r}/r_0 + i r_0 \, \mathbf{p})/\sqrt{2}$, $\mathbf{b}^+ = (\mathbf{r}/r_0 - i r_0 \, \mathbf{p})/\sqrt{2}$ are the annihilation and creation operators, respectively, and $r_0 = \sqrt{\hbar/m\omega}$ is the characteristic oscillator length. There exists another representation [2] for the single-particle transformation operator,

$$U(\mathbf{r}, \mathbf{p}, \boldsymbol{\sigma}) = (d_0 \, d_0^+)^{-1/2} \, d_0, \tag{3.4}$$

which can be obtained from Eq. (3.2) by applying the identity

$$\hat{x} \, f(\hat{y}\hat{x}) = f(\hat{x}\hat{y}) \, \hat{x}.$$

The latter identity requires the property of associativity for the operators $\hat{x}$ and $\hat{y}$ for its proof, and holds for any analytic function $f(\hat{z})$ which is expandable in a power series.

The transformation operator, as given by Eqs.(3.2) and (3.4), acts on the harmonic oscillator eigenstates (2.4) in the following manner [17, 13]

$$U(\mathbf{r}, \mathbf{p}, \boldsymbol{\sigma}) \, \psi_{nljj_z} = \psi_{\tilde{n}\tilde{l}jj_z}, \tag{3.5}$$

where n is the number of quanta, j is the angular momentum, l and m are the orbital momentum and its projection, $\mathbf{Y}_l$ is a spherical harmonic, and χ is a Pauli

spinor. The "pseudo" values of n and l are determined by the rules

$$\tilde{n} = n - 1; \;\; \tilde{l} = l \pm 1 \, \text{if} \, j = l \pm 1/2. \tag{3.6}$$

The normal→pseudo transformation is rotationally invariant and unitary. Rotational invariance follows from the fact that the angular momentum $\mathbf{j} = \mathbf{l} + \mathbf{s}$ commutes with the d and d^+ operators. Unitarity holds within the subspace of normal parity orbitals only, that is, within the space spanned by the set of states of a major shell less the one with $(j = n + 1/2)$. The unique parity orbitals, which either defect out of the model space $(j = n + 1/2)$ or intrude into it from the shell above $(j = n + 3/2)$ due to the spin-orbit interaction, have no pseudo counterparts because they are annihilated by the d operator (cf. r.h.s. of Eq. (3.4)). For instance, the $s_{1/2}$, $d_{3/2}$, $d_{5/2}$ and $g_{7/2}$ orbitals of the $n = 4$ shell map onto the $\tilde{p}_{1/2}$, $\tilde{p}_{3/2}$, $\tilde{f}_{5/2}$ and $\tilde{f}_{7/2}$ orbitals of the $\tilde{n} = 3$ shell, respectively, while the $g_{9/2}$ orbital has no pseudo image. The unitarity can be checked by comparing Eqs.(3.2) and (3.4) with the condition

$$U_0 U_0^+ = U_0^+ U_0 = 1. \tag{3.7}$$

For the sake of notational simplicity the transformation operator is denoted by U_0 in what follows. The subcript implies the spherical limit of the oscillator shell model and is used throughout this chapter for consistency with the following one.

3.2 Permutation relations

The definitions (3.2) and (3.4) clearly indicate that the rotational invariants (3.3) are the simplest building blocks of the normal→pseudo transformation operator. As shown in Ref. [3], these two operators are also the odd generators of the osp(1|2)

superalgebra. When combined with the bilinear forms of d_0 and d_0^+, namely,

$$\begin{aligned} d_0^2 &= \mathbf{b}\cdot\mathbf{b}, \\ h_{osc} &= \frac{1}{2}(d_0\, d_0^+ + d_0^+ d_0) = n + \frac{3}{2}, \\ (d_0^+)^2 &= \mathbf{b}^+\cdot\mathbf{b}^+, \end{aligned} \tag{3.8}$$

they form an algebra closed with respect to commutation and anticommutation operations (see Eqs.(3.1) and (4.3) from [3] for the relations connecting the osp(1|2) generators). The n symbol is used henceforth for denoting the number of quanta operator.

Since the operators (3.3) and (3.8) are related through both commutations and anticommutation, they should also be involved in general permutation relations. Indeed, by using the well-known rule,

$$(\boldsymbol{\alpha}\cdot\boldsymbol{\sigma})(\boldsymbol{\beta}\cdot\boldsymbol{\sigma}) = (\boldsymbol{\alpha}\cdot\boldsymbol{\beta}) + i\,\boldsymbol{\sigma}\cdot(\boldsymbol{\alpha}\times\boldsymbol{\beta}), \tag{3.9}$$

and the standard commutation relations for $\mathbf{b}$ and $\mathbf{b}^+$, it is easy to see that

$$\begin{aligned} (\mathbf{b}\cdot\boldsymbol{\sigma})(\mathbf{l}\cdot\boldsymbol{\sigma}) &= -(\mathbf{l}\cdot\boldsymbol{\sigma} + 2)(\mathbf{b}\cdot\boldsymbol{\sigma}), \\ (\mathbf{b}\cdot\boldsymbol{\sigma})\,n &= (n+1)(\mathbf{b}\cdot\boldsymbol{\sigma}). \end{aligned} \tag{3.10}$$

Here $\mathbf{l} = i\,\mathbf{b}\times\mathbf{b}^+$ is the orbital angular momentum. An iterative use of Eq. (3.10) leads to the permutation rule,

$$(\mathbf{b}\cdot\boldsymbol{\sigma})\, g(n, \mathbf{l}\cdot\boldsymbol{\sigma}) = g(n+1, -\mathbf{l}\cdot\boldsymbol{\sigma} - 2)\,(\mathbf{b}\cdot\boldsymbol{\sigma}), \tag{3.11}$$

which is valid for operator-valued functions of the two variables. The operators n and $\mathbf{l}\cdot\boldsymbol{\sigma}$ naturally appear in this expression because they are simple linear combinations of dd^+ and d^+d (see Eqs.(3.2), (3.4), (3.15) and (3.16)). A similar formula can be given for the d^+ operator:

$$(\mathbf{b}^+\cdot\boldsymbol{\sigma})\, g(n, \mathbf{l}\cdot\boldsymbol{\sigma}) = g(n-1, -\mathbf{l}\cdot\boldsymbol{\sigma} - 2)\, (\mathbf{b}^+\cdot\boldsymbol{\sigma}). \tag{3.12}$$

Note that Eqs.(3.11) and (3.12) can be folded into the same relation

$$(\mathbf{b}^\pm\cdot\boldsymbol{\sigma})\, g(n, \mathbf{l}\cdot\boldsymbol{\sigma}) = g(n \mp 1, -\mathbf{l}\cdot\boldsymbol{\sigma} - 2)\, (\mathbf{b}^\pm\cdot\boldsymbol{\sigma}) \tag{3.13}$$

provided $\mathbf{b}^-$ stands for $\mathbf{b}$.

The permutation rule (3.13) is a cornerstone of the procedure for developing analytical results for pseudo transforms. It is of primary use for transforming monopole operators, that is, operators that are rotational scalars. In the case of higher multipolarity operators there are more complex rules which are not reducible to permutations only. The rules are different for different operators, but generally, the degree of complexity increases rapidly with the multipolarity of the operator. For instance, the rule for transposing the spin operator with an analytic function of the $\mathbf{l}\cdot\boldsymbol{\sigma}$ operator goes as follows:

$$\boldsymbol{\sigma} f(\mathbf{l}\cdot\boldsymbol{\sigma}) = f(-\mathbf{l}\cdot\boldsymbol{\sigma} - 1)\,\boldsymbol{\sigma} + \frac{f(\mathbf{l}\cdot\boldsymbol{\sigma}) - f(-\mathbf{l}\cdot\boldsymbol{\sigma} - 1)}{\mathbf{l}\cdot\boldsymbol{\sigma} + 1/2}\,\mathbf{j}, \tag{3.14}$$

where the symbolic operator-valued fraction is used because the numerator and denominator do commute. This formula is derived in the Appendix B and will be used in Sec. 3.5 where the image of the spin operator is given.

3.3 DOUBLE TRANSFORMATION

A simple, although rather interesting, application of Eq. (3.11) is a derivation of an analytic expression for the twofold normal→pseudo transformation. An important property of the double transformation is that it actually depends only on the orbital degrees of freedom. This property will prove useful for finding the transform of the spin operator (see Sec. 3.5).

To obtain the double transformation result, note that

$$d_0^+ d_0 = n - \mathbf{l}\cdot\boldsymbol{\sigma}, \tag{3.15}$$

$$d_0\, d_0^+ = n + \mathbf{l}\cdot\boldsymbol{\sigma} + 3, \tag{3.16}$$

and rewrite the single transformation (see Eqs.(3.2) and (3.4)) in the form [13]

$$U_0 = (\mathbf{b}\cdot\boldsymbol{\sigma})(n - \mathbf{l}\cdot\boldsymbol{\sigma})^{-1/2}, \tag{3.17}$$

or, equivalently [2],

$$U_0 = (n + \mathbf{l}\cdot\boldsymbol{\sigma} + 3)^{-1/2}(\mathbf{b}\cdot\boldsymbol{\sigma}). \tag{3.18}$$

Note that Eq. (3.11) provides an additional and direct proof of the identity between operators (3.17) and (3.18) acting in the normal parity subspace.

The double transformation can now formally be defined as a product of two single transformations. For instance, Eq. (3.18) yields

$$U_0^2 = (n + \mathbf{l}\cdot\boldsymbol{\sigma} + 3)^{-1/2}(\mathbf{b}\cdot\boldsymbol{\sigma})(n + \mathbf{l}\cdot\boldsymbol{\sigma} + 3)^{-1/2}(\mathbf{b}\cdot\boldsymbol{\sigma}).$$

By applying Eq. (3.11), the $\mathbf{b}\cdot\boldsymbol{\sigma}$ operator can be moved to the right resulting in

$$U_0^2 = \left((n+2)(n+3) - \mathbf{l}^2\right)^{-1/2} \mathbf{b}^2. \tag{3.19}$$

To arrive at (3.19), the identities

$$(\mathbf{l}\cdot\boldsymbol{\sigma})^2 = \mathbf{l}^2 - \mathbf{l}\cdot\boldsymbol{\sigma}, \tag{3.20}$$

$$(\mathbf{b}\cdot\boldsymbol{\sigma})^2 = \mathbf{b}^2 \tag{3.21}$$

were used along with the fact that the n and $\mathbf{l}\cdot\boldsymbol{\sigma}$ operators commute. Since

$$(n+2)(n+3) - \mathbf{l}^2 = \mathbf{b}^2(\mathbf{b}^+)^2,$$

Eq. (3.19) can be rewritten as

$$U_0^2 = \left(\mathbf{b}^2(\mathbf{b}^+)^2\right)^{-1/2} \mathbf{b}^2. \tag{3.22}$$

Thus, the double normal→pseudo transformation is reduced to an action of the $\mathbf{b}^2$ operator with a subsequent normalization (cf. Eq. (3.5)):

$$U_0^2 \psi_{nljj_z} = \psi_{n-2,ljj_z}. \tag{3.23}$$

From an algebraic viewpoint, this transformation can be expressed in terms of the enveloping algebra of the noncompact symplectic algebra sp(2,R) which is a subalgebra of osp(1|2). The three generators of sp(2,R) are defined in Eq. (3.8). It is important to note, however, that while the sp(2,R) algebra that emerges is related to the sp(2,R) subalgebra of the nuclear collective motion algebra of sp(6,R)

[26, 43], in general these two algebras only coincide at the single-particle level. In the many-particle case the generators of the collective sp(2,R) algebra include a summation over single-particle operators (3.8). In the pseudospin-related problems only single-particle operators are considered; for instance, the many-particle double transformation is just a product of single-particle transformations (3.22). This product structure of $U^2_{0,total}$ is an indication of the *non-collective* nature of the pseudospin transformation.

Note that in contrast with the single transformation, the double form changes neither the orbital momentum nor parity — it is an O(3) scalar operator. As a result, the spin is also invariant with respect to the double transformation. In short, the double transformation carries the n of the oscillator into $n-2$ while leaving both l and s unchanged.

The technique used for deriving Eqs.(3.19) and (3.22) can be used to produce another form of the double transformation operator

$$\begin{aligned} U^2_0 &= \mathrm{b}^2 \left(n(n+1) - \mathrm{l}^2\right)^{-1/2} \\ &= \mathrm{b}^2 \left((\mathrm{b}^+)^2 \mathrm{b}^2\right)^{-1/2}. \end{aligned} \tag{3.24}$$

Which form is used in an application is simply a matter of convenience so long as the consideration is confined to the normal parity subspace.

3.4 One-body rotational invariants

One-body rotationally invariant operators naturally emerge in microscopic, shell-model based approaches to nuclear structure. For pseudospin-related problems the d_0, d_0^+ operators and the single-nucleon Hamiltonian are the characteristic rota-

tional scalars. The normal→pseudo transforms of these operators are derived in this section.

Analytic results for the transformed d_0 and d_0^+ operators, which are an integral part of the normal→pseudo transformation itself, follow from the definition of a transformed operator,

$$F' = U_0 F U_0^+, \tag{3.25}$$

where F' is the transform of F. For the $F = d_0 = \mathbf{b}\cdot\boldsymbol{\sigma}$ case, it is convenient to utilize Eq. (3.18) for U and the Hermitian conjugate of Eq. (3.17) for U_0^+. Applying formula (3.16) to this result yields

$$U_0 \mathbf{b}\cdot\boldsymbol{\sigma} U_0^+ = \left(\frac{n - \mathbf{l}\cdot\boldsymbol{\sigma} + 2}{n + \mathbf{l}\cdot\boldsymbol{\sigma} + 3}\right)^{1/2} \mathbf{b}\cdot\boldsymbol{\sigma}. \tag{3.26}$$

The transform of $d_0^+ = \mathbf{b}^+\cdot\boldsymbol{\sigma}$ is given by the Hermitian conjugate of Eq. (3.26). For convenience of having the normalization factor on the left, the rule (3.12) can be applied to obtain the result

$$U_0 \mathbf{b}^+\cdot\boldsymbol{\sigma} U_0^+ = \left(\frac{n + \mathbf{l}\cdot\boldsymbol{\sigma} + 3}{n - \mathbf{l}\cdot\boldsymbol{\sigma}}\right)^{1/2} \mathbf{b}^+\cdot\boldsymbol{\sigma}. \tag{3.27}$$

Given the transformation properties of d and d^+, it becomes easy to transform the single-nucleon Hamiltonian for the oscillator shell model (in units of $\hbar\omega$),

$$H = h_{osc} - k\left(\mathbf{l}\cdot\boldsymbol{\sigma} + \mu(\mathbf{l}^2 - \langle\mathbf{l}^2\rangle_n)\right), \tag{3.28}$$

where h_{osc} is the oscillator energy operator (see (3.8)) and $\langle\mathbf{l}^2\rangle_n = n(n+3)/2$ is the mean value of $\mathbf{l}^2$ within the n-th shell. The $\langle\mathbf{l}^2\rangle_n$ term is subtracted from $\mathbf{l}^2$ to ensure that the average value of the single-nucleon Hamiltonian remains fixed by h_{osc} [29].

The transformation of the oscillator energy is obvious,

$$U_0 h_{osc} U_0^+ = h_{osc} + 1, \tag{3.29}$$

because U_0 reduces the number of oscillator quanta by 1. To understand how the spin-orbit term transforms, recall relations (3.15) and (3.16). Since the transforms for d, d^+ and h_{osc} are known, it is easy to show that

$$U_0 \mathbf{l}\cdot\boldsymbol{\sigma} U^+ = -\mathbf{l}\cdot\boldsymbol{\sigma} - 2. \tag{3.30}$$

And finally, by making use of Eq. (3.20), the transform of $\mathbf{l}^2$ can be determined:

$$U_0 \mathbf{l}^2 U_0^+ = \mathbf{l}^2 + 2\mathbf{l}\cdot\boldsymbol{\sigma} + 2. \tag{3.31}$$

Since $\boldsymbol{\sigma}^2$ has a unique eigenvalue it is invariant under the normal→pseudo transformation, and furthermore, Eqs.(3.30) and (3.31) provide proof for the invariance of $\mathbf{j}^2 = \mathbf{l}^2 + \mathbf{l}\cdot\boldsymbol{\sigma} + \boldsymbol{\sigma}^2/4$. The latter result is simply a consequence of the rotational invariance of the normal→pseudo transformation itself.

By combining the results of Eqs.(3.29–3.31), the transformation of the Hamiltonian (3.28) can be given,

$$\begin{aligned} U_0 H U_0^+ = h_{osc} + 1 \;&-\; k\left((2\mu - 1)\mathbf{l}\cdot\boldsymbol{\sigma} + \mu(\mathbf{l}^2 - \langle \mathbf{l}^2 \rangle_n)\right) \\ &+\; k\left(\mu(h_{osc} - \frac{3}{2}) + 2\right). \end{aligned} \tag{3.32}$$

This expression for the transformed Hamiltonian coincides with the corresponding formula in Ref.[13], with the exception of the $\langle \mathbf{l}^2 \rangle_n$ term and its transform which

were not considered in the earlier pseudospin studies. (Note that the inclusion of the $\langle \mathbf{l}^2 \rangle_n$ term in the original Hamiltonian induces a slight change in the oscillator frequency of the transformed Hamiltonian because the value of $k\mu$ is about 0.02–0.04 for heavy nuclei.) Also, there is no easy way to apply the commutator technique employed in [13] to operators whose transforms have a more complicated form, for example, (3.26) and (3.27) as well as many other operators of physical significance. The techniques based on rule (3.13) and its generalizations are applicable in all cases.

The expressions derived so far are sufficient for calculating the transforms for any polynomial (or more complex) functions of the d_0 and d_0^+ operators. Important examples of this kind are the bilinear combinations $d_0^2 = \mathbf{b}^2$ and $(d_0^+)^2 = (\mathbf{b}^+)^2$ which together with the n operator generate the sp(2,R) algebra (see comment following Eq. (3.8)). Their transforms can be found by squaring both sides of Eqs.(3.26) and (3.27) and applying rule (3.13):

$$U_0 \mathbf{b}^2 U_0^+ = \left(\frac{n + \mathbf{l}\cdot\boldsymbol{\sigma} + 5}{n + \mathbf{l}\cdot\boldsymbol{\sigma} + 3} \right)^{1/2} \mathbf{b}^2, \tag{3.33}$$

$$U_0 (\mathbf{b}^+)^2 U_0^+ = \left(\frac{n + \mathbf{l}\cdot\boldsymbol{\sigma} + 3}{n + \mathbf{l}\cdot\boldsymbol{\sigma} + 1} \right)^{1/2} (\mathbf{b}^+)^2. \tag{3.34}$$

An obvious application of the above results is a calculation of the transform of the monopole transition operator $\mathbf{r}^2$ which is a linear combination of the three symplectic generators (3.8).

3.5 Spin and quadrupole moment operators

While rotational scalars transform in a rather simple manner, the transformation of the higher multipolarity operators requires a more advanced prescription. Below,

such a transformation is developed for the spin and Elliott quadrupole operator (the latter is that part of the quadrupole moment operator which conserves the number of oscillator quanta). These operators are important for applications because their matrix elements enter into expressions for moments and transition rates. The transformation for the orbital momentum is also found since the total angular momentum $\mathbf{j}$ is known to be invariant under the normal→pseudo transformation.

By using definition (3.17), the image of the spin operator can be written in the form

$$U_0\boldsymbol{\sigma}U_0^+ = (\mathbf{b}\cdot\boldsymbol{\sigma})(n-\mathbf{l}\cdot\boldsymbol{\sigma})^{-1/2}\boldsymbol{\sigma}(n-\mathbf{l}\cdot\boldsymbol{\sigma})^{-1/2}(\mathbf{b}^+\cdot\boldsymbol{\sigma}). \tag{3.35}$$

Now recall Eq. (3.14) to discover that

$$\begin{aligned}(n-\mathbf{l}\cdot\boldsymbol{\sigma})^{-1/2}\boldsymbol{\sigma}\ (n-\mathbf{l}\cdot\boldsymbol{\sigma})^{-1/2} &= [(n+\mathbf{l}\cdot\boldsymbol{\sigma}+1)(n-\mathbf{l}\cdot\boldsymbol{\sigma})]^{-1/2}\,\boldsymbol{\sigma} \\ &+\frac{(n-\mathbf{l}\cdot\boldsymbol{\sigma})^{-1}-[(n+\mathbf{l}\cdot\boldsymbol{\sigma}+1)(n-\mathbf{l}\cdot\boldsymbol{\sigma})]^{-1/2}}{\mathbf{l}\cdot\boldsymbol{\sigma}+1/2}\,\mathbf{j}.\end{aligned} \tag{3.36}$$

By inserting the r.h.s. of the latter expression in the r.h.s. of Eq. (3.35) and applying the permutation rule, the following expression for the transformed spin operator is obtained:

$$U_0\boldsymbol{\sigma}U_0^+ = \frac{(\mathbf{b}\cdot\boldsymbol{\sigma})\boldsymbol{\sigma}(\mathbf{b}^+\cdot\boldsymbol{\sigma})+2\left[1+\left(\frac{n-\mathbf{l}\cdot\boldsymbol{\sigma}}{n+\mathbf{l}\cdot\boldsymbol{\sigma}+3}\right)^{1/2}\right]^{-1}\mathbf{j}}{[(n+\mathbf{l}\cdot\boldsymbol{\sigma}+3)(n-\mathbf{l}\cdot\boldsymbol{\sigma})]^{1/2}}. \tag{3.37}$$

The use of symbolic division in this formula is justified because the operators that enter into both the numerator and denominator factors commute with one another. The fact that the $(\mathbf{b}\cdot\boldsymbol{\sigma})\boldsymbol{\sigma}(\mathbf{b}^+\cdot\boldsymbol{\sigma})$ operator commutes with the denominator, follows

as a particular case from an identity,

$$[(\mathbf{b}\cdot\boldsymbol{\sigma})\sigma(\mathbf{b}^{+}\cdot\boldsymbol{\sigma}),\, G(n,\mathbf{l}^2+\mathbf{l}\cdot\boldsymbol{\sigma}+2)]=0, \tag{3.38}$$

which is valid for an operator-valued analytical function $G(x,y)$ of the two variables. This result follows from the fact that

$$[\boldsymbol{\sigma},\, G(n,\mathbf{l}^2)]=0 \tag{3.39}$$

by applying the $\mathbf{b}\cdot\boldsymbol{\sigma}$ and $\mathbf{b}^{+}\cdot\boldsymbol{\sigma}$ operators on the left and right, respectively.

Equation (3.37) is one of several equivalent forms for the transformed spin operator. The equality

$$U_0^{+}\boldsymbol{\sigma}U_0=\frac{(\mathbf{b}^{+}\cdot\boldsymbol{\sigma})\sigma(\mathbf{b}\cdot\boldsymbol{\sigma})-2\left[1+\left(\frac{n+\mathbf{l}\cdot\boldsymbol{\sigma}+3}{n-\mathbf{l}\cdot\boldsymbol{\sigma}}\right)^{1/2}\right]^{-1}\mathbf{j}}{[(n+\mathbf{l}\cdot\boldsymbol{\sigma}+3)(n-\mathbf{l}\cdot\boldsymbol{\sigma})]^{1/2}}, \tag{3.40}$$

which can be derived in a similar manner, is another form for the same expression because of the invariance of the spin under the double normal→pseudo transformation. By taking an average of the r.h.s. of Eqs.(3.37) and (3.40) and using the identity

$$(\mathbf{b}\cdot\boldsymbol{\sigma})\sigma_i(\mathbf{b}^{+}\cdot\boldsymbol{\sigma})+(\mathbf{b}^{+}\cdot\boldsymbol{\sigma})\sigma_i(\mathbf{b}\cdot\boldsymbol{\sigma})=4q_{ij}\sigma^j-\frac{2}{3}(n+\frac{3}{2})\sigma_i\,, \tag{3.41}$$

where a summation over repeated indices is implied and

$$q_{ij}=\frac{1}{2}(b_i b_j^{+}+b_j b_i^{+})-\frac{1}{3}(n+3)\delta_{ij} \tag{3.42}$$

are the Cartesian components of the Elliott quadrupole tensor, the transform of the spin operator can be re-expressed in terms of the spin, orbital momentum and quadrupole single-particle operators only:

$$U_0\sigma U_0^+ = [(n+1\cdot\sigma+3)(n-1\cdot\sigma)]^{-1/2} \tag{3.43}$$
$$\times\left(2(\mathrm{q}\otimes\sigma)^{(1)} - \frac{1}{3}(n+\frac{3}{2})\sigma + \frac{3\mathrm{j}}{\left[(n+1\cdot\sigma+3)^{1/2}+(n-1\cdot\sigma)^{1/2}\right]^2}\right).$$

In this expression $(\mathrm{q}\otimes\sigma)_i \equiv q_{ij}^{(1)}\sigma^j$. Although the resulting expression looks more complicated than any of the monopole operator transforms discussed in the previous section, evaluation of the corresponding matrix elements poses no problem in the spherical oscillator single-particle basis.

Proceeding to a derivation of the transform of the Elliott quadrupole tensor, it is convenient to start from the definition

$$U_0\mathrm{q}U_0^+ = (n+1\cdot\sigma+3)^{-1/2}(\mathrm{b}\cdot\sigma)\mathrm{q}(\mathrm{b}^+\cdot\sigma)(n+1\cdot\sigma+3)^{-1/2}. \tag{3.44}$$

By utilizing Eqs.(3.42) and (3.16), the boson commutation relations, and the definition of the orbital momentum, it is possible to prove the identity

$$(\mathrm{b}\cdot\sigma)q_{ij}(\mathrm{b}^+\cdot\sigma) \tag{3.45}$$
$$= \frac{1}{2}\left\{q_{ij}(n+1\cdot\sigma+4) + (n+1\cdot\sigma+4)q_{ij} + \frac{1}{3}1\cdot\sigma\delta_{ij} - \frac{1}{2}(l_i\sigma_j + l_j\sigma_i)\right\}$$

and rewrite the previous equation as follows

$$U_0\mathrm{q}U_0^+ = \frac{1}{2}(n+1\cdot\sigma+3)^{-1/2}$$

$$\times\{\mathbf{q}(n+1\cdot\sigma+4)+(n+1\cdot\sigma+4)\mathbf{q}-(1\otimes\sigma)^{(2)}\} \qquad (3.46)$$
$$\times(n+1\cdot\sigma+3)^{-1/2},$$

where $(1\otimes\sigma)^{(2)}_{ij}=(l_i\sigma_j+l_j\sigma_i-(1\cdot\sigma)\delta_{ij})/3$ stands for the spherical tensor of rank 2 formed out of the orbital and spin momenta. Rewriting Eq. (3.46) as a fraction of commuting operators as was done for Eqs.(3.37) and (3.40) leads to no obvious advantage as the resulting expression is neither transparent nor particularly convenient for applications. Moreover, as will be discussed in the next section, the present form is well suited for an analysis which reveals the principal components of the tensorial structure of its image and which can be easily generalized in the many-particle case.

3.6 Approximate pseudospin transforms

A comparison of the results of the previous two sections confirms that the pseudospin transforms of higher multipolarity operator forms are more complicated than those for monopole operators. This is especially true for Hermitian forms that conserve the total number of oscillator quanta.

It is interesting to juxtapose the analytical results from above for the single-particle case with many-particle SU(3)-tensorial expansions for multipole operators determined numerically using the relabeling procedure referred to above together with standard group-theoretical coupling techniques [42]. The results show that the dominant parts of these seemingly complex operators have a relatively simple structure that in each case is very close to the structure of the original operator.

For example, the transformed spin operator has the analytical form

$$U_{0,total}\mathbf{S}U^{+}_{0,total} = -\frac{1}{3}\mathbf{S} + \left(\sum_{i=1}^{A} \boldsymbol{\lambda}_i^{(l=2)} \otimes \mathbf{s}_i\right)^{(J=1)}, \tag{3.47}$$

where $\boldsymbol{\lambda}^{(l=2)}$ is an orbital operator of the quadrupole type. The SU(3) tensorial expansion of the $\boldsymbol{\lambda}$ operator, obtained in Ref. [42], consists of a leading term proportional to the Elliott quadrupole operator with the rest of the series not reducible to SU(3) generators but adding up to a very small part of the total value of the calculated matrix elements. The coefficients in this expansion are oscillator-shell dependent. (A further transformation of Eq. (3.43) also yields Eq. (3.47), although in a tedious and nontransparent way.)

For the Elliott quadrupole tensor as well as for the spherical rank 0 and 2 tensors which are Sp(6,R) generators that increase/decrease the number of oscillator quanta by two, the transformation rule is even simpler:

$$U_{0,total}FU^{+}_{0,total} = \kappa_F F + \cdots, \tag{3.48}$$

where ellipsis represents other SU(3) tensors that have expansion coefficients which are less than ten percent of the leading term and tend to effectively cancel on average so as to yield less than one percent change in calculated transition rates [10, 12]. The coefficients κ are usually operator and shell dependent with the latter dependence decreasing monotonically towards unity with increasing shell number.

Given the simple form for the leading term in these expansions, it seems reasonable to expect that the analytical techniques developed for the single-particle case should lead to an easy way of predicting the structure of the dominant parts of a transformed operator as well as a prescription for evaluating the corresponding

expansion coefficients. An approximate method for doing this is proposed below; however, caution is advised as there is no simple method short of a full calculation for giving an estimate for errors that might be associated with the use of such approximations.

The procedure is based on the following observation: In general, for a given single-particle operator F there exists several different pairs of operators $\breve{F}$ and G satisfying the identity

$$d_0 F d_0^+ = \frac{1}{2}(d_0 d_0^+ \breve{F} + \breve{F} d_0 d_0^+) + G \tag{3.49}$$

with d_0 and d_0^+ defined in Eq. (3.3). Different choices for $\breve{F}$ and G are possible because the operator F may be encountered not only in commutation relations with d_0 and d_0^+ but also in the anticommutation and generic permutation relations (see Sec. 3.3). While $\breve{F}$ usually has a tensorial structure similar to F, the structure of the residual term G is dependent upon the choice of the permutation relation that is used in the derivation. In what follows, the choice that renders the structure of G as simple as possible is made, namely, that choice which involves a minimum number of SU(3) tensor operators.

The transformed operator can be written in the form

$$\begin{aligned} U_0 F U_0^+ &= \frac{1}{2}\left((d_0 d_0^+)^{1/2}\breve{F}(d_0 d_0^+)^{-1/2} + (d_0 d_0^+)^{-1/2}\breve{F}(d_0 d_0^+)^{1/2}\right) \\ &\quad +(d_0 d_0^+)^{-1/2} G (d_0 d_0^+)^{-1/2}. \end{aligned} \tag{3.50}$$

Note that within a given oscillator shell the $(d_0 d_0^+)^{-1/2} = (n + 1\cdot\sigma + 3)^{-1/2}$ factor, as well as its inverse, is a positive definite, monotonic, and slowly changing function of the $\mathbf{l}\cdot\boldsymbol{\sigma}$ operator, especially for higher shells. Since the pseudospin symmetry is

relevant for heavy nuclei and high single-particle orbitals, it is not unreasonable to approximate the r.h.s. of Eq. (3.50) by taking average values within the shell for both factors, and these in turn can be estimated by setting $\mathbf{l}\cdot\boldsymbol{\sigma} \to 0$ (or in a better way, if possible).

It is important to recall, however, that average values of the single-particle angular momenta and quadrupole moments within a given major shell correlate with the shell number. So while a formal expansion in powers of $\mathbf{l}\cdot\boldsymbol{\sigma}$, which is the basis of the subsequent consideration, apparently is asymptotic, the result remains approximate and should be used with appropriate caution.

The $\left((d_0 d_0^+)^{1/2}\breve{F}(d_0 d_0^+)^{-1/2} + (d_0 d_0^+)^{-1/2}\breve{F}(d_0 d_0^+)^{1/2}\right)/2$ operator of Eq. (3.50) can be approximated by $\breve{F}$. This is appropriate because the two operators behave similarly under Hermitian conjugation, have the same traces in any subspace of single-particle states, and their difference can only be on the order of $O(n^{-2})$. The latter estimate is valid because of the absence of a linear $\mathbf{l}\cdot\boldsymbol{\sigma}$ term in the MacLaurin series for $\left((d_0 d_0^+)^{1/2}\breve{F}(d_0 d_0^+)^{-1/2} + (d_0 d_0^+)^{-1/2}\breve{F}(d_0 d_0^+)^{1/2}\right)/2$. For the residual $(d_0 d_0^+)^{-1/2}G(d_0 d_0^+)^{-1/2}$ term the estimate $\beta_F(n+3)^{-1}G$ is acceptable with β_F an adjustable parameter that is close to unity. This parameter accounts for higher-order corrections due to averaging and renormalization and can be evaluated directly or by comparison with known results.

These considerations lead to the following approximation

$$U_0 F U_0^+ = \breve{F} + \frac{\beta_F}{n+3}G + O(n^{-2}) \tag{3.51}$$

for the transform of the operator F. The accuracy of this approximation is expected to increase with increasing shell number. Obviously, such an approximation is not

unique, and there is always a chance to improve it by using a more sophisticated initial expression. For instance, as will be demonstrated below, Eq. (3.43) allows for immediate averaging without any preliminary transformation.

As examples, three cases from the previous sections, namely the transformation of the $\mathbf{l}\cdot\boldsymbol{\sigma}$, $\boldsymbol{\sigma}$ and $\mathbf{q}$ operators, will now be considered. The result for $F = \mathbf{l}\cdot\boldsymbol{\sigma}$ is particularly simple because in this case $\breve{F} = -(\mathbf{l}\cdot\boldsymbol{\sigma} + 2)$ and $G = 0$ by virtue of Eq. (3.10), (3.11) or (3.12). In this case the exact result, (3.30), is obtained as a consequence of the commutation of the dd^+ and $\breve{F}$ operators.

For $F = \boldsymbol{\sigma}$ there is no need to apply the generic scheme based on Eq. (3.49) because it is more convenient to average the values of the slowly changing coefficients in the r.h.s. of Eq. (3.43). If accuracy is maintained to $O(n^{-2})$, the approximate transform for the spin operator $\mathbf{s} = \boldsymbol{\sigma}/2$ has the form

$$U_0 \mathbf{s} U_0^+ = -\frac{1}{3}\mathbf{s} + \frac{2\beta_s}{n+3/2}(\mathbf{q}\otimes\mathbf{s})^{(1)} + \cdots \tag{3.52}$$

which can be compared directly with its many-particle generalization (3.47). This shows that the coefficient of the spin operator is exact, and the SO(3) tensorial structure is represented correctly. Moreover, as mentioned above, the SU(3) tensorial expansion for $\boldsymbol{\lambda}^{(l=2)}$, found in [42], shows that the dominant term has the transformation properties of the Elliott quadrupole operator. Therefore, these two approaches are in a good agreement. An estimate for the coefficient β_s from Eq. (3.43) is simply unity. As an alternative, Eq. (3.52) can be raised to the second power and then if the expression were exact the r.h.s. should be equal to 3/4. In general, the result depends on both the $\mathbf{l}\cdot\boldsymbol{\sigma}$ and $\mathbf{l}^2$ operators; however, the value of $\beta_s = \sqrt{8/5}$ guarantees the correct average value for $\mathbf{s}^2$ within a shell. These two estimates are

very close, with the differences attributable to corrections that are on the order of n^{-2} and higher which are effectively taken into account in the latter estimate.

To determine the dominant part of the transform for the single-particle Elliott quadrupole operator $\mathbf{q}$, compare Eq. (3.46) to the definition (3.50) and note that in this case $\breve{F} = \mathbf{q}$ and $G = \mathbf{q} - (\mathbf{l} \otimes \mathbf{s})^{(2)}$. Then by making use of prescription (3.51) this can be rewritten as

$$U_0 \mathbf{q} U_0^+ = \mathbf{q} + \frac{\beta_q}{n+3} \left(\mathbf{q} - (\mathbf{l} \otimes \mathbf{s})^{(2)} \right) + \cdots \tag{3.53}$$

Although there is no rigorous way to evaluate β_q, the following heuristic estimate proves to be rather precise. By comparing the corresponding terms from Eqs.(3.46) and (3.53), the ratio $\beta_q/(n+3)$ within the given shell can be shown to be close to the average value of the $(n + \mathbf{l}\cdot\boldsymbol{\sigma} + 3)^{-1}$ operator within the same shell. Using the formal expansion,

$$\frac{n+3}{n+\mathbf{l}\cdot\boldsymbol{\sigma}+3} = 1 - \frac{\mathbf{l}\cdot\boldsymbol{\sigma}}{n+3} + \frac{\mathbf{l}^2 - \mathbf{l}\cdot\boldsymbol{\sigma}}{(n+3)^2} + O(n^{-3}), \tag{3.54}$$

and familiar formulas $\langle \mathbf{l}\cdot\boldsymbol{\sigma} \rangle_n = 0$ and $\langle \mathbf{l}^2 \rangle_n = n(n+3)/2$ for the average values within the n-th oscillator shell, the following approximation is obtained:

$$\beta_q \approx 1 + \frac{n}{2(n+3)}.$$

Note that the comparison of the "empirical" relation (3.48) with Eq. (3.53) displays the simple connection

$$\kappa_q = 1 + \frac{\beta_q}{n+3}.$$

To illustrate the accuracy of the result, compare the estimates for κ_q, calculated according to this formula, to the exact numerical values from Ref.[10]. The relevant numbers are 1.208 vs 1.221 for $n = 3$, 1.184 vs 1.193 for $n = 4$, and 1.164 vs 1.171 for $n = 5$, respectively. The difference is about one per cent and decreasing, *i.e.* the accuracy is apparently higher than can be expected from the rough estimates given above.

The occurrence of the residual term proportional to $(\mathrm{l} \otimes \mathrm{s})^{(2)}$, that is predicted by Eq. (3.53), is also corroborated by the results given in Ref.[10]. Indeed, the SU(3)⊃SO(3) tensorial expansion of the many-particle image of **Q** contains a term with the structure of $(\sum_{i=1}^{A} \mathrm{l}_i \otimes \mathrm{s}_i)^{(J=2)}$ although its influence on the E2 transition rates is weak compared to that of the leading term.

3.7 Concluding remarks

A general prescription for generating *normal→pseudo* transforms of physical operators in the context of a spherical harmonic oscillator shell-model theory has been introduced. The procedure applies to operators that can be expressed in terms of single-particle oscillator boson operators $\mathbf{b}$, $\mathbf{b}^+$ (or coordinates $\mathbf{r}$ and momenta $\mathbf{p}$) and spins $\mathbf{s}$, and is based on the existence of permutation relations among the rotational invariants constructed out of $\mathbf{b}$, $\mathbf{b}^+$ and $\mathbf{s}$. A simple and straightforward consequence of these permutation relations is the existence of an analytical result for the double transformation in terms of rotational scalars bilinear in $\mathbf{b}$ and $\mathbf{b}^+$ that form the noncompact sp(2,R) symplectic algebra.

Although the pseudospin representation has been widely used in the past, the option of applying it in an analytical form adds a new dimension to the many-particle studies of the structure of heavy deformed nuclei. In contrast to the general

and powerful but formal algebraic technique, using the SU(3)⊃SO(3) tensorial expansions plus relabeling of the single-particle states, the procedure of analytical transformation allows to derive the *normal→pseudo* images for the operators expressed in customary physical variables.

The application of the transformation procedure to different operators yields results of different complexity. For instance, the transforms for the number of oscillator quanta, spin-orbit and orbit-orbit terms from the spherical Nilsson Hamiltonian are very simple and already well known [9, 13, 17]. The transforms of other rotational scalars are slightly more complicated. The operators of higher multipolarity tend to have images that are not reducible to any simple or transparent form, and arriving at the exact final expressions usually requires some creative thought.

Fortunately, for cases of real physical interest the exactness of the *normal→pseudo* transformation can be easily compromised in favor of relative simplicity. Indeed, by using the appropriate permutation relation and averaging over slowly varying operator-valued factors within a given oscillator shell, it is feasible to extract the leading part of the transform which has a simple structure and accurately approximates the entire operator. These approximations can be used to streamline applications of the theory by rendering it no more complicated than the usual physical representation while reducing the spin-orbit interaction in the mean-field and the space of states to the normal parity subspace only. Representative operators for which an approximate form has proven to be advantageous include the electromagnetic transition operators and the multipole interactions which are of high significance for the studies of the collective phenomena in heavy nuclei.

A noteworthy aspect of the normal→pseudo transformation is its underlying algebraical structure. The results of Secs. 3.3–3.6 underscore the significance of the

connection between this transformation and the orthosymplectic supersymmetry and its subsymmetries.

Once the normal→pseudo transformation is expressed in terms of the osp(1|2) superalgebra [3] of the rotational invariants of $\mathbf{b}$, $\mathbf{b}^+$ and $\mathbf{s}$, it is natural to expect that the transforms of these rotational invariants are themselves expressible in terms of the same superalgebra, and this fact is demonstrated explicitly. The double transformation is shown to separate the orbital and spin variables and therefore can be expressed in terms of the sp(2,R) Lie algebra which is the subalgebra of osp(1|2). The double transforms of the bilinear rotational invariants (3.8) are, in turn, written in terms of sp(2,R). The inclusion of the nonscalar bilinear combinations would extend the dynamical symmetry algebra to sp(6,R).

The normal→pseudo transforms of the spin and orbital momenta and the Elliott quadrupole operator are expressible in terms of these same three operators in the normal space. The $\mathbf{l}$ and $\mathbf{q}$ operators form the Lie algebra of the Elliott SU(3) group [22], an important subgroup of the Sp(6,R) group. Therefore, in the single-particle case the normal→pseudo transformation produces an automorphism of the universal enveloping algebra of the SU(3)⊗SU(2) group. This algebraic property is no longer exact for the many-particle operators $\mathbf{L}$,$\mathbf{Q}$ and $\mathbf{S}$, which comprise the collective su(3)⊕su(2) algebra. Nevertheless, the dominant parts of the pseudospin transforms of these operators are known to be proportional to the operators themselves. This is apparently the reason why the corresponding operators in the many-particle pseudospace are well defined, and this in turn leads to the pseudo-SU(3) and pseudo-Sp(6,R) models being valid physical theories of the collective phenomena in heavy nuclei.

CHAPTER 4

PSEUDOSPIN AND NUCLEAR DEFORMATION

Once the explicit form of the transformation operator is known, one can learn more about the origin of the pseudospin symmetry and check its goodness directly by transforming the Hamiltonian of an appropriate nuclear model. This test is preferable to the analysis of the single-particle spectra since the latter does not provide evidence for the weakness of spin-orbit interaction in the presence of various other forces. If the spin-orbit strength in the transformed Hamiltonian is strongly reduced for any deformation, one can speak about the pseudospin dynamical symmetry.

The study of the goodness of the pseudospin dynamical symmetry in triaxial nuclei within the framework of the harmonic oscillator shell model is the subject of this chapter. In what follows, an explicit form of the extended pseudospin transformation for arbitrary deformations is suggested and applied to some modifications of the triaxial Nilsson Hamiltonian.

It is worth recalling that the shell-model pseudospin representation, which is adapted to the oscillator basis, has a precursor emerging in realistic mean-field and many-particle descriptions of nuclei. The realistic nuclear models are shown to require the momentum space helicity transformation for the effective reduction in the spin-orbit interaction strength and the normal→pseudo relabeling of the spin and orbital momenta [6]. However, the helicity transforms of the oscillator wavefunctions are no longer the oscillator eigenfunctions, especially in the nuclear surface region where they acquire different asymptotics. The pseudospin transformation of the spherical oscillator functions can therefore be understood as the helicity trans-

formation followed by a special projection back onto the oscillator basis. This operational definition is taken in this chapter, generalized for the triaxial case and used as a foundation for the subsequent analysis.

The discussion starts from a review of model single-particle Hamiltonians and their pseudo transforms in the spherical and asymptotic prolate cases. Based on an explicit introduction of the triaxially deformed pseudospin operator and an approximate analytical evaluation of its action on the components of the model Hamiltonians, the functional form and spectra of the transformed Hamiltonians are analyzed. This analysis validates the microscopic "pseudo" models on both the single-particle and many-particle levels.

4.1 Deformation-dependent pseudospin transformation

The natural choice of the mean field for harmonic oscillator related nuclear studies is provided by the Nilsson model and its extensions [29, 40] which amend the deformed oscillator Hamiltonian so as to reproduce the observed shell structure.

In two limiting cases of the Nilsson model, the spherical and strongly prolate deformed oscillator, there exist well established analytically solutions. In these two cases appropriate forms for the pseudospin transformation were found [13, 15] which allow for the transformation of the Hamiltonian in a closed analytical form. This is not accidental and is indicative of the fact that there exists an intrinsic connection between the model Hamiltonian and the corresponding transformation. As shown in the current section, this line of thought leads to an explicit form of the deformation-dependent pseudospin transformation which happens to be closely related to some

Hamiltonians of the Nilsson type and simultaneously to the helicity transformation considered in the previous section.

A generic Nilsson-type Hamiltonian (in units of $\hbar\omega$) for a triaxially deformed nucleus can be written in the form

$$h = h_{osc} - k(u_{ls} + \mu u_{ll}), \tag{4.1}$$

where h_{osc} denotes the anisotropic (deformed) harmonic oscillator,

$$h_{osc} = \sum_s \frac{p_s^2 + (m\omega_s x_s)^2}{2m\hbar\omega} = \sum_s \epsilon_s \left(\mathsf{b}_s^+ \mathsf{b}_s + \frac{1}{2}\right), \tag{4.2}$$

with $s = z, x, y$ denoting the three Cartesian axes, and u_{ls} and u_{ll} symbolizing the spin-orbit and orbit-orbit interactions, respectively, whose structure is specified below for each of the cases. The values of the dimensionless parameters k and μ are determined mainly by the mass region of the nucleus. The boson operators are defined in the deformed basis by

$$\mathsf{b}_s = \sqrt{\frac{m\omega_s}{2\hbar}} x_s + \frac{i}{\sqrt{2\hbar m\omega_s}} p_s, \quad \mathsf{b}_s^+ = \sqrt{\frac{m\omega_s}{2\hbar}} x_s - \frac{i}{\sqrt{2\hbar m\omega_s}} p_s \tag{4.3}$$

and obey the standard commutation relations

$$[\mathsf{b}_r, \mathsf{b}_s^+] = \delta_{rs}, \quad [\mathsf{b}_r, \mathsf{b}_s] = [\mathsf{b}_r^+, \mathsf{b}_s^+] = 0 \tag{4.4}$$

(note the different font used throughout the chapter for the operators in the deformed basis). The dimensionless frequencies $\epsilon_s = \omega_s/\omega$ are subject to volume conservation

$$\epsilon_z \epsilon_x \epsilon_y = 1. \tag{4.5}$$

4.1.1 SPHERICAL LIMIT

In the spherical case ($\epsilon_z = \epsilon_x = \epsilon_y = 1$),

$$h_{osc} \rightarrow n + \frac{3}{2},\ u_{ls} \rightarrow \mathbf{l}\cdot\boldsymbol{\sigma},\ u_{ll} \rightarrow \mathbf{l}^2 - \langle \mathbf{l}^2 \rangle_n, \tag{4.6}$$

where the nondeformed number of quanta operator $n = \mathbf{b}^+ \cdot \mathbf{b}$ and the physical orbital momentum $\mathbf{l} = i\,\mathbf{b}\times\mathbf{b}^+$ are constructed out of the spherical boson operators

$$b_s = \sqrt{\frac{m\omega}{2\hbar}}x_s + \frac{i}{\sqrt{2\hbar m\omega}}p_s,\ b_s^+ = \sqrt{\frac{m\omega}{2\hbar}}x_s - \frac{i}{\sqrt{2\hbar m\omega}}p_s. \tag{4.7}$$

The mean value of $\mathbf{l}^2$ over a given shell, $\langle \mathbf{l}^2 \rangle_n = n(n+3)/2$, is subtracted to ensure that the average single-nucleon energy within a shell is fixed by its harmonic oscillator value [29].

The eigenstates $|nljj_z\rangle$ of the Hamiltonian, specified by Eqs. (4.1) and (4.6), in the coordinate representation are given by formula (2.4), and the corresponding eigenvalues are easily calculable:

$$\varepsilon_{nlj} = n + \frac{3}{2} + k\left(2(l-j)\left(l+\frac{1}{2}\right) + \frac{1}{2} - \mu\left(l(l+1) - \frac{1}{2}n(n+3)\right)\right). \tag{4.8}$$

The single-particle pseudospin operator in this limit can be written in the explicit form [13]

$$U_0 = d_0\,(d_0^+\,d_0)^{-1/2}, \tag{4.9}$$

or, equivalently [2],

$$U_0 = (d_0\,d_0^+)^{-1/2}\,d_0, \tag{4.10}$$

where

$$d_0 = \mathbf{b} \cdot \boldsymbol{\sigma}. \tag{4.11}$$

The equivalence between Eqs. (4.9) and (4.10) and the unitarity property for the spherical pseudospin transformation,

$$U_0 U_0^+ = U_0^+ U_0 = 1, \tag{4.12}$$

follow from the operator-valued identity,

$$\hat{x}\, f(\hat{y}\hat{x}) = f(\hat{x}\hat{y})\, \hat{x}, \tag{4.13}$$

which requires the property of associativity of the $\hat{x}$ and $\hat{y}$ operators for its proof, and holds for any analytic function $f(\hat{z})$ expandable in a power series. Since the structure of the pseudospin transformation for an arbitrary anisotropy of the oscillator field is similar to the structure of U_0 (see the following subsections), both the equivalent forms of the transformation operator and the unitarity property are also valid in the most general case.

When acting onto the entire space of basis states, the U_{sph} operator projects out the subspace of defector states ($j = n+1/2$) and performs the unitary transformation of the normal parity states subspace in the following manner

$$U_0 \, |nljj_z\rangle = |\tilde{n}\tilde{l}jj_z\rangle, \tag{4.14}$$

where the relabeling rule $\tilde{n} = n-1, \tilde{l} = 2j-l$ is the same as discussed in the previous section.

The rules of the pseudospin transformation for the operators under consideration are known [5, 13]

$$U_0\, n\, U_0^+ = n + 1, \tag{4.15}$$

$$U_0\, \mathbf{l}\cdot\boldsymbol{\sigma}\, U_0^+ = -\mathbf{l}\cdot\boldsymbol{\sigma} - 2, \tag{4.16}$$

$$U_0\, \mathbf{l}^2\, U_0^+ = \mathbf{l}^2 + 2\mathbf{l}\cdot\boldsymbol{\sigma} + 2, \tag{4.17}$$

and yield the image for the Hamiltonian in the form

$$U_0\, h\, U_0^+ = h_{osc} + 1 - k\left((2\mu - 1)u_{ls} + \mu u_{ll}\right) + 2k(1-\mu) + k\mu(n+2), \tag{4.18}$$

where the h_{osc}, u_{ls} and u_{ll} terms are formally the same as in (4.6) but now they act in the pseudospin representation. The transformed Hamiltonian, which is known as the pseudo Nilsson one [44], has a structure close to the original Hamiltonian structure. A significant difference between the two Hamiltonians is a sharp reduction in the spin-orbit splitting since the empirical average value of μ is almost 0.5 (more precisely, it is about 0.4 for neutrons and 0.6 for protons [17]). This reduction is the cornerstone of the pseudospin symmetry. Another distinction occurs because of the last term in Eq.(4.18); it produces an insignificant increase in the oscillator frequency because the value of $k\mu$ is in the 0.02–0.04 interval for heavier nuclei.

As a consequence of the unitarity, the pseudo Nilsson spherical Hamiltonian is isospectral to the Nilsson Hamiltonian in the normal parity sector; namely, formula (4.8) yields the energy of both $|nljj_z\rangle$ and $|\tilde{n}\tilde{l}jj_z\rangle$ states.

A noteworthy property of the spherical pseudospin transformation (as well as of the helicity one) is its rotational invariance, *i.e.* the conservation of the angular

momentum operator $\mathbf{j} = \mathbf{l}+\mathbf{s}$. The proof follows from the commutation rule $[\mathbf{j}, \mathbf{b}\sigma] = 0$ and the Hermiticity of $\mathbf{j}$.

4.1.2 CYLINDRICAL LIMIT

Since the Nilsson Hamiltonian is based on a phenomenological description of the nuclear mean field rather than a rigorous derivation from a more microscopic model, there is no unambiguous prescription for an explicit form of the u_{ls} and u_{ll} operators for deformed nuclei. Nevertheless, it is usually assumed following Nilsson's arguments [40] that total number of quanta $\mathsf{n} = \sum_s \mathsf{b}_s^+ \mathsf{b}_s$ in the deformed basis is well conserved, and the u_{ls} and u_{ll} terms for the strong deformation region have to be chosen accordingly. (Even the conventional Nilsson scheme, which is applicable at average deformations and uses the spherical assumption (4.6) for these terms, is known to preserve n within several percent accuracy up to the superdeformed region. However, at higher deformations the mixing among various deformed shells becomes essential.)

In the case of axial deformation ($\epsilon_x = \epsilon_y = \epsilon$, $\epsilon_z = 1/\epsilon^2$), the model Hamiltonian is no longer rotationally invariant and commutes with j_z only. An increase in ϵ makes a nuclear shape more prolate, and at some point the asymptotic Nilsson limit is reached where the number of longitudinal quanta n_z is conserved. If the model Hamiltonian conserves n, as suggested above, then regardless of its exact form the spectrum is actually determined by its part commuting with n_z, and the Hamiltonian terms can be effectively written as

$$h_{osc} \to \epsilon(\mathsf{n}_\rho + 1) + \frac{1}{\epsilon^2}\left(\mathsf{n}_z + \frac{1}{2}\right), \; u_{ls} \to \zeta l_z \sigma_z, \; u_{ll} \to \xi\left(l_z^2 - \langle l_z^2 \rangle_{\mathsf{n}}\right), \qquad (4.19)$$

where ζ and ξ are the model parameters, $\langle l_z^2 \rangle_n = n(n+3)/6$ is the trace-equivalent part of l_z^2 within the deformed shell, and $n_\rho = n_x + n_y$ is the number of transverse quanta. (Note that l_z coincides with l_z in the axial case.) The eigenstates of the effective Hamiltonian, determined by Eqs. (4.1) and (4.19), are the Nilsson asymptotic states in the cylindrical basis $|nn_z l_z j_z\rangle$ with $j_z = l_z + 1/2$ or $l_z - 1/2$.

The Nilsson diagrams in the deformed region clearly indicate that, although the spin-orbit strength is large, there exist some pairs of levels whose energy splitting is weak and slowly decreasing with deformation. In the domain of strong deformation, these pairs consist of the asymptotic orbitals $|n, n_z, l_z, l_z + 1/2\rangle$ and $|n, n_z, l_z + 2, l_z + 3/2\rangle$.

As shown in Ref. [15], this fact has an explanation if the pseudospin transformation in the asymptotic region is chosen in the form

$$U_\infty = d_\infty \, (d_\infty^+ \, d_\infty)^{-1/2}, \tag{4.20}$$

where

$$d_\infty = b_x \sigma_x + b_y \sigma_y. \tag{4.21}$$

This transformation is unitary and its equivalent form

$$U_\infty = (d_\infty \, d_\infty^+)^{-1/2} \, d_\infty \tag{4.22}$$

holds similarly to Eq. (4.10).

The U_∞ transformation conserves j_z, the longitudinal component of the angular momentum, and acts on the cylindrical basis states according to the rule

$$U_\infty \left| \mathsf{n}, \mathsf{n}_z, l_z, l_z \pm \frac{1}{2} \right\rangle = \left| \mathsf{n}, \mathsf{n}_z, l_z \pm 1, l_z \pm \frac{1}{2} \right\rangle . \tag{4.23}$$

This rule is equivalent to the relabeling procedure

$$\tilde{\mathsf{n}} = \mathsf{n} - 1, \ \tilde{\mathsf{n}}_z = \mathsf{n}_z, \ \tilde{l}_z = l_z + 2s_z, \ \tilde{s}_z = -s_z \tag{4.24}$$

within the subspace of cylidrical states with $\mathsf{n}_\rho > l_z$. The pairs of nearly degenerate states $|\mathsf{n}, \mathsf{n}_z, l_z, l_z + 1/2\rangle$ and $|\mathsf{n}, \mathsf{n}_z, l_z + 2, l_z + 3/2\rangle$ are then relabeled as $|\tilde{\mathsf{n}}, \tilde{\mathsf{n}}_z, \tilde{l}_z, l_z - 1/2\rangle$ and $|\tilde{\mathsf{n}}, \tilde{\mathsf{n}}_z, \tilde{l}_z, l_z + 1/2\rangle$, respectively, and can be treated as pseudospin doublets. The rest of the states, with $\mathsf{n}_\rho = l_z$, span the subspace where the action of U_∞ is undefined.

Under the U_∞ transformation, the operators entering Eq. (4.19) acquire the following images

$$U_\infty \, \mathsf{n}_\rho \, U_\infty^+ = \mathsf{n}_\rho + 1, \tag{4.25}$$

$$U_\infty \, \mathsf{n}_z \, U_\infty^+ = \mathsf{n}_z, \tag{4.26}$$

$$U_\infty \, l_z \, \sigma_z U_\infty^+ = -l_z \sigma_z - 1, \tag{4.27}$$

$$U_\infty \, l_z^2 \, U_\infty^+ = l_z^2 + 2 l_z \sigma_z + 1, \tag{4.28}$$

which naturally are in accordance with the relabeling rules. Then the transformed Hamiltonian can be written in the form

$$U_\infty \, h \, U_\infty^+ = h_{osc} + \epsilon - k \left(\left(2\frac{\zeta}{\xi}\mu - 1 \right) u_{ls} + \mu u_{ll} \right) + k\zeta(1-\mu) + \frac{1}{3} k\mu\zeta(\mathsf{n}+2). \tag{4.29}$$

Therefore, similarly to the spherical limit, in the cylindrical limit the Nilsson-type Hamiltonian can be replaced by its "pseudo" version which is characterized by much weaker spin-orbit splitting and slightly higher oscillator frequencies (provided the value of ζ/ξ is close to 1). As mentioned above, the action of the asymptotic pseudo Nilsson Hamiltonian is confined to the subspace of the entire space of states but this subspace is different from the normal parity subspace of the spherical limit.

4.1.3 Generic case

As seen from Eqs. (4.9), (4.10) and (4.20), (4.22), the structure of the pseudospin transformation operator in the spherical and cylindrical limits is very similar although the basic structural blocks (4.11) and (4.21) are somewhat different. It is natural to assume that the d_0 and d_∞ operators are the limiting cases of an operator d which is the structural block of a generic pseudospin transformation

$$\begin{aligned} U &= d\,(d^+\,d)^{-1/2} \\ &= (d\,d^+)^{-1/2}\,d. \end{aligned} \tag{4.30}$$

The latter transformation is unitary and valid for arbitrary deformation.

To find out the explicit form of the d operator, one can assume that the pseudospin transformation is closely related to the helicity one. Indeed, if the helicity transformation plays the same part in the realistic nuclear structure models as the pseudospin transformation does in the oscillator shell model, there should exist a direct connection between the two transformations.

Note that the single-particle helicity transformation $\mathcal{H}$ can also be written in the form (4.30) with the structural block

$$d_{hel} = \mathbf{p} \cdot \boldsymbol{\sigma}. \tag{4.31}$$

The helicity operation is a universal transformation acting on the microscopic level regardless of any specific properties of a given nucleus. However, when the discussion is confined to a mean-field description, one can model the self-consistent field with the deformed oscillator, determined by the frequencies ϵ_z, ϵ_x and ϵ_y, and henceforth rewrite Eq. (4.31) as

$$d_{hel} = -i\sqrt{\frac{2}{\hbar m\omega}}\sum_s \sqrt{\epsilon_s}(\mathsf{b}_s - \mathsf{b}_s^+)\sigma_s, \tag{4.32}$$

where the boson operators are defined by Eq. (4.3). Now note that the d_{hel} operator mixes the states of a deformed shell with the states from the shell below as well as the shell above. To comply with requirements on the pseudospin operator, considered in section 4.1, one must exclude interactions with the shell above. Also, the constant factor in the definition of d_{hel} can be ignored since it has no effect on the U operator according to Eq. (4.30). Finally, the structural block d can be defined as follows:

$$d = \sum_s \sqrt{\epsilon_s}\mathsf{b}_s\sigma_s. \tag{4.33}$$

This equation together with Eq. (4.30) uniquely defines the deformation-dependent pseudospin transformation.

At this point it is expedient to formulate several general requirements which guarantee internal consistency of the entire approach and its relevance for the nuclear structure calculations:

1. Correspondence with the known limiting cases;

2. Conformity between the Hamiltonian (4.1) and the transformation at any given deformation. In more detail,

   - the transformation does not destroy the shell structure of the deformed oscillator basis,
   - the transformed Hamiltonian is isospectral to the original one in the regular sector of the entire space of states which is defined as the kernel (the region of unitarity) of the transformation,
   - the operators entering both the Hamiltonian and the transformation are closely related;

3. Equivalence of the spherical and deformed pseudo representations for physical applications. The deformed pseudo Nilsson Hamiltonian must be the same whether it is constructed by inserting the deformation into the spherical pseudo Nilsson Hamiltonian (4.18) or by applying the generic transformation to the original Hamiltonian (4.1). This condition implies that an explicit form of the u_{l_s} and u_{ll} operators is chosen.

Checking and implementation of these requirements determine the structure and contents of the rest of the chapter. The first two requirements will be considered in this subsection. The last one needs an analytical expression for the transformed Hamiltonian and will be discussed in the last section in more detail.

It is evident that in the spherical limit the operator (4.33) coincides with (4.9) and (4.10). In the axial case ($\epsilon_x = \epsilon_y = \epsilon$, $\epsilon_z = 1/\epsilon^2$) the d operator is reduced to $d_{ax} = \mathsf{b}_x\sigma_x + \mathsf{b}_y\sigma_y + \epsilon^{-3/2}\mathsf{b}_z\sigma_z$ after the elimination of the common factor of $\sqrt{\epsilon}$. The asymptotic prolate shape formally corresponds to the limit $\epsilon \gg 1$ in which case d_{ax} becomes very close to d_∞, and the results [15], reviewed in the previous subsection, can be reproduced. Nevertheless, this limiting case should be treated with caution. Indeed, the ratio of frequencies $\omega_x/\omega_z = \epsilon^3$ is known to be close to 2 in the superdeformation region, and therefore the contribution of $\mathsf{b}_z\sigma_z$ is not negligible in the experimentally achievable domain.

The shell structure of the anisotropic oscillator is preserved by the transformation (4.30), (4.33) since the shell number gets decreased exactly by one: $\bar{\mathsf{n}} = \mathsf{n} - 1$. However, in contrast to the spherical and cylindrical limits, the Hamiltonian eigenstates in the generic case do not coincide with the eigenstates of the harmonic oscillator. The pseudospin transformation divides the nth deformed oscillator shell in the normal representation into two regions which can be named a regular sector R_n and a singular one S_n. Within the regular sector, which is analogous to the normal parity subspace in the spherical limit, the transformation is unitary. The singular sector is similar to the defector subspace. It is defined as the part of the space of states which is annihilated by the d operator,

$$d\,\mathsf{S}_\mathsf{n} = 0, \tag{4.34}$$

and within this sector the transformation is undetermined. The transformation maps the region of unitarity onto the oscillator shell with one quantum less, and the latter shell in the pseudo representation becomes representative of the whole

former shell in the normal representation. The eigenstates within the former and the latter shells can be determined only through the action of the original and transformed Hamiltonians, respectively. Within the region of unitarity, both the Hamiltonians are necessarily isospectral, and there exists a one-to-one mapping between the corresponding eigenstates.

A noteworthy fact, although not emphasized in Refs. [13, 15], is a close relation between the structural block of the transformation and the operators entering the Hamiltonian. By comparing the relations

$$n + \frac{3}{2} = \frac{1}{2}[d_0, d_0^+]_+, \tag{4.35}$$

$$\mathbf{l}\cdot\boldsymbol{\sigma} + \frac{3}{2} = \frac{1}{2}[d_0, d_0^+], \tag{4.36}$$

$$n_\rho + 1 = \frac{1}{2}[d_\infty, d_\infty^+]_+, \tag{4.37}$$

$$l_z\sigma_z + 1 = \frac{1}{2}[d_\infty, d_\infty^+], \tag{4.38}$$

$$\mathbf{l}^2 = (\mathbf{l}\cdot\boldsymbol{\sigma})^2 + \mathbf{l}\cdot\boldsymbol{\sigma}, \tag{4.39}$$

$$l_z^2 = (l_z\sigma_z)^2, \tag{4.40}$$

where $[\hat{x}, \hat{y}]_+$ denotes an anticommutator, with Eqs. (4.6), (4.19), note that in both the limiting cases the harmonic oscillator and spin-orbit splitting terms are bilinear combinations of d and d^+, and the orbit-orbit interaction is a square of the spin-orbit splitting operator plus a correction eliminating the dependence on the spin variables. (The only exclusion is the boson number operator whose transformation poses no problem.)

It is natural to suggest that similar relations are valid in the generic case. Indeed, the harmonic oscillator term (4.2) is expressible as

$$h_{osc} = \frac{1}{2}[d, d^+]_+. \tag{4.41}$$

Since an adequate formula for the spin-orbit splitting in the strong deformation domain is not yet established, it is expedient to pursue the above analogy further and make the following assumption:

$$\begin{aligned} u_{ls} &= \frac{\zeta}{2}([d, d^+] - m_1) \\ &= \zeta \boldsymbol{\Lambda}\cdot\boldsymbol{\sigma}, \end{aligned} \tag{4.42}$$

where ζ is the model parameter which may depend on deformation. The Cartesian components of the $\boldsymbol{\Lambda}$ vector,

$$\Lambda_s = \frac{l_s}{\sqrt{\epsilon_s}}, \tag{4.43}$$

are defined through the components l_s of the orbital momentum of the deformed representation, and the m_q parameters are given by the formula

$$m_q = \sum_s \epsilon_s^q \tag{4.44}$$

for any real q. Note also that Eq. (4.42) uses the volume conservation condition.

Following this line of thought, choose the orbit-orbit interaction operator in the form

$$u_{ll} = \xi(\boldsymbol{\Lambda}^2 - \langle \boldsymbol{\Lambda}^2 \rangle_n), \tag{4.45}$$

$$(\Lambda^2)_n = \frac{m_{-1}}{6} n(n+3). \tag{4.46}$$

By introducing an auxiliary operator Λ' with components

$$\Lambda'_s = \sqrt{\epsilon_s}\, l_s, \tag{4.47}$$

rewrite Λ^2 in the form

$$\Lambda^2 = (\Lambda \cdot \sigma)^2 + \Lambda' \cdot \sigma, \tag{4.48}$$

reminiscent of Eqs. (4.39) and (4.40).

It is easily seen that the u_{ls} and u_{ll} operators thus obtained fit both the spherical and cylindrical limits and perform a natural extrapolation to other deformation regions. However, despite this smooth extrapolation, an analytical transform for the model Hamiltonian cannot be written in a closed form. The argument behind this statement is purely algebraic. Note that in the spherical limit, d_0, d_0^+ and their bilinear combinations form a closed set under both the commutation and anticommutation relations known as the osp(1|2) superalgebra [3]. The closure of the permutation relations is the principal reason why the transformed Nilsson Hamiltonian becomes almost as simple as the original one. The same situation holds in the cylindrical limit. In Appendix C it is proven that the operator $\sum_s f(\epsilon_s) b_s \sigma_s$, where $f(x)$ is an analytic function, its Hermitian conjugate and, therefore, any combination of such operators can be written in terms of some combination of the d and d^+ operators only. Thus, generally speaking, the above defined operators entering the generic Nilsson-type Hamiltonian, as well as their pseudospin transforms, can still be rewritten in terms of d and d^+ only. However, in contrast to the spherical and cylindrical limits, the permutation relations no longer close for a finite deformation,

and such a rewrite cannot be folded into a finite simple expression and thus would be of no help for practical purposes.

Nevertheless, there exists an analytical, although approximate, solution to the problem which is quite reasonable for higher shells, *i.e.* in the case of the heavy nuclei. It is based on a technique that proved rather accurate in the spherical representation for the operators not reducible to combinations of d_0 and d_0^+. This technique and the resulting approximate transforms for the Hamiltonian terms are discussed in the next section.

4.2 Approximate pseudospin transforms

Consider the permutation rule

$$d\, f(d^+ d) = f(d\, d^+)\, d$$

which is a particular case of Eq. (4.13). By applying this rule along with the definitions (4.30), (4.33) of the transformation operator, one obtains the formula

$$U d^+ d\, U^+ = d\, d^+, \tag{4.49}$$

which can also be written as

$$U \left(h_{osc} - \boldsymbol{\Lambda}\cdot\boldsymbol{\sigma} - \frac{m_1}{2} \right) U^+ = h_{osc} + \boldsymbol{\Lambda}\cdot\boldsymbol{\sigma} + \frac{m_1}{2}. \tag{4.50}$$

This result and the obvious equality,

$$U \mathsf{n} U^+ = \mathsf{n} + 1, \tag{4.51}$$

are apparently the only independent pseudospin transforms derivable in exact analytical form. For instance, a similar rule for the d and $f(d\,d^{+})$ operators does not exist as a consequence of the identity,

$$d\,(d\,d^{+}) = (d^{+}d)\,d + 2\sum_{s} \epsilon_{s}^{3/2} \mathrm{b}_{s}\sigma_{s}, \tag{4.52}$$

and the fact that $\sum_{s} \epsilon_{s}^{3/2}\mathrm{b}_{s}\sigma_{s}$ is not a linear function of $d = \sum_{s} \epsilon_{s}^{1/2}\mathrm{b}_{s}\sigma_{s}$, except for the spherical and asymptotic axial ($\epsilon \gg 1$ or $\epsilon \ll 1$) limits. (Note that Eq. (4.52) is just a rewrite of the commutation relation,

$$\left[\sum_{s} \epsilon_{s}\mathrm{b}_{s}^{2}, \sum_{s} \epsilon_{s}^{1/2}\mathrm{b}_{s}^{+}\sigma_{s}\right] = 2\sum_{s} \epsilon_{s}^{3/2}\mathrm{b}_{s}\sigma_{s},$$

with the help of the auxiliary relation $\sum_{s} \epsilon_{s}\mathrm{b}_{s}^{2} = d^{2}$.)

The above notes imply that the exact transform of the Hamiltonian (4.1) might look very cumbersome. Nevertheless, given the similarity between the original and transformed Hamiltonians in both the limiting cases considered in the previous section, it seems reasonable to expect that there exists an analytical but approximate procedure for calculating the Nilsson-type Hamiltonian image for any deformation which yields a transform of the pseudo Nilsson type. This kind of procedure is proposed and discussed below. The errors associated with the use of such approximations can be roughly evaluated *a priori* in a simple analytical form; a rigorous *a posteriori* numerical estimate is provided by the isospectrality condition.

4.2.1 Approximation procedure

The procedure is based on the following observation: In general, for a given single-particle operator F there exist several different pairs of operators $\breve{F}$ and G satisfying

the identity,

$$dFd^+ = \frac{1}{2}[\breve{F}, dd^+]_+ + G, \tag{4.53}$$

with d and d^+ defined in Eq. (4.33). Different choices for $\breve{F}$ and G are possible because the d and d^+ operators contain the spin variables that in turn enables applying not only commutation relations but also anticommutation and generic permutations to the l.h.s. of Eq. (4.53). While $\breve{F}$ usually has a structure similar to F, the structure of the residual term G is dependent upon the choice of the permutation relation that is used in the derivation. In what follows, it is assumed that the permutation relation is chosen in order to ultimately simplify the structure of G.

The transformed operator can therefore be written in the form

$$\begin{aligned} UFU^+ &= \frac{1}{2}\left((dd^+)^{1/2}\breve{F}(dd^+)^{-1/2} + (dd^+)^{-1/2}\breve{F}(dd^+)^{1/2}\right) \\ &\quad +(dd^+)^{-1/2}G(dd^+)^{-1/2}. \end{aligned} \tag{4.54}$$

Note that the $(d\,d^+)^{-1/2} = (h_{osc} + \boldsymbol{\Lambda}\cdot\boldsymbol{\sigma} + m_1/2)^{-1/2}$ and its inverse, $(d\,d^+)^{1/2}$, are regular operator-valued functions within a given oscillator shell. Their common argument can be represented within the nth shell as

$$d\,d^+ = \langle d\,d^+\rangle_{\mathsf{n}} + \sum_s(\epsilon_s\,\delta\mathsf{n}_s + \Lambda_s\sigma_s), \tag{4.55}$$

where

$$\langle d\,d^+\rangle_{\mathsf{n}} = m_1\frac{\mathsf{n}+3}{3} \tag{4.56}$$

is the average value of the operator, and $\delta\mathsf{n}_s = \mathsf{n}_s - \mathsf{n}/3$ is the deviation of the number of bosons along a Cartesian axis from the mean value.

Since the pseudospin symmetry is relevant for heavy nuclei and high single-particle orbitals, it makes sense to approximate the r.h.s. of Eq. (4.54) by making a formal operator-valued expansion with $(n+3)^{-1}$ as a small parameter. The advantage of such an expansion is that the mean value within the shell of the first-order correction often vanishes, and in some physically interesting cases the correction itself vanishes. It is important to recall, however, that average values of the single-particle angular momenta within the shell correlate with the shell number. So the above formal expansion is apparently asymptotic and should be used with appropriate caution.

The expansion technique is applied below only to those operators which are bilinear in boson operators. When formally rewritten in series of inverse powers of $n+3$, such operators may contain only negative first and zeroth degree terms. Therefore, it is reasonable to neglect the impact of the terms of the positive first and higher degrees. The approximate transforms thus obtained carry no explicit dependence on the expansion parameter, *i.e.* they are shell-independent. As shown later, they provide the exact results in both the spherical and cylindrical limits.

The $\left((dd^+)^{1/2}\breve{F}(dd^+)^{-1/2}+(dd^+)^{-1/2}\breve{F}(dd^+)^{1/2}\right)/2$ operator of Eq. (4.54) can be approximated by $\breve{F}$. This is appropriate because the two operators behave similarly under Hermitian conjugation, have the same traces in any subspace of single-particle states, and their difference can only be on the order of $O((n+3)^{-2})$. The latter estimate is valid since the linear term vanishes in the expansion for $\left((dd^+)^{1/2}\breve{F}(dd^+)^{-1/2}+(dd^+)^{-1/2}\breve{F}(dd^+)^{1/2}\right)/2$.

The G operator is representable in the form

$$G=\frac{n+3}{3}G_{-1}+G_0 \tag{4.57}$$

which estimates the residual $(dd^+)^{-1/2}G(dd^+)^{-1/2}$ term by G_{-1}/m_1. Obviously, Eqs. (4.55) and (4.56) display a particular case of the representation (4.57).

By summing the above expressions, the approximation

$$UFU^+ = \breve{F} + \frac{1}{m_1}G_{-1} + O(\frac{1}{\mathsf{n}+3}) \tag{4.58}$$

for the transform of the operator F is obtained. The $\breve{F}$ and G_{-1} operators are defined in Eqs. (4.53) and (4.57), and the dominant part of the error in the approximation occurs due to the latter operator. The accuracy of the approximation is expected to increase with increasing shell number. It is also noteworthy that the procedure of averaging over the oscillator shell, which is in the foundation of Eq. (4.58), is quite natural for the Nilsson-type models since these models normally use the values of parameters fixed for the nuclei within given shells.

4.2.2 Scalar operators

Approximate transforms for the h_{osc}, $h'_{osc} = \sum_s \epsilon_s^{-1}(\mathsf{n}_s + 1/2)$, $\boldsymbol{\Lambda}\cdot\boldsymbol{\sigma}$, $\mathsf{l}\cdot\boldsymbol{\sigma}$, and $\boldsymbol{\Lambda}'\cdot\boldsymbol{\sigma}$ operators, where l is the orbital momentum vector of the deformed representation whose Cartesian components are l_s, $s = x, y, z$, can be obtained by applying Eq. (4.58). Strictly speaking, it is more than sufficient for finding the transform of the Hamiltonian defined by Eqs. (4.1), (4.2) and (4.42–4.48). Nevertheless, having this set of operators determined allows for a generalization of the result to similar operators with an arbitrary deformation dependence.

By using the boson commutation relations (4.3) and the well-known multiplication rule

$$(\boldsymbol{\sigma}\cdot\mathbf{u})(\boldsymbol{\sigma}\cdot\mathbf{v}) = i\,\boldsymbol{\sigma}\cdot(\mathbf{u}\times\mathbf{v}), \tag{4.59}$$

it is possible to check the validity of the identities:

$$d\,h_{osc}\,d^{+} = \frac{1}{2}\left([h_{osc}, dd^{+}]_{+} + \sum_{s}\epsilon_{s}^{3/2}\left(\mathsf{b}_{s}\sigma_{s}\,d^{+} + d\,\mathsf{b}_{s}^{+}\sigma_{s}\right)\right), \tag{4.60}$$

$$d\,h'_{osc}\,d^{+} = \frac{1}{2}\left([h'_{osc}, dd^{+}]_{+} + \sum_{s}\epsilon_{s}^{-1/2}\left(\mathsf{b}_{s}\sigma_{s}\,d^{+} + d\,\mathsf{b}_{s}^{+}\sigma_{s}\right)\right), \tag{4.61}$$

$$d\,\boldsymbol{\Lambda}\cdot\boldsymbol{\sigma}\,d^{+} = \frac{1}{2}\left(-[2m_{1} + \boldsymbol{\Lambda}\cdot\boldsymbol{\sigma}, dd^{+}]_{+} + \sum_{s}\epsilon_{s}^{3/2}\left(\mathsf{b}_{s}\sigma_{s}\,d^{+} + d\,\mathsf{b}_{s}^{+}\sigma_{s}\right)\right), \tag{4.62}$$

$$\begin{aligned} d\,\mathsf{l}\cdot\boldsymbol{\sigma}\,d^{+} = \frac{1}{2}\Bigg([2 - \mathsf{l}\cdot\boldsymbol{\sigma}, dd^{+}]_{+} + 2\sum_{s}\epsilon_{s}^{1/2}\left(\mathsf{b}_{s}\mathsf{l}_{s}\,d^{+} + d\,\mathsf{b}_{s}^{+}\mathsf{l}_{s}\right) \\ - m_{1/2}\sum_{s}\left(\mathsf{b}_{s}\sigma_{s}\,d^{+} + d\,\mathsf{b}_{s}^{+}\sigma_{s}\right)\Bigg), \end{aligned} \tag{4.63}$$

$$\begin{aligned} d\,\boldsymbol{\Lambda}'\cdot\boldsymbol{\sigma}\,d^{+} = \frac{1}{2}\Bigg(-[\boldsymbol{\Lambda}'\cdot\boldsymbol{\sigma}, dd^{+}]_{+} + 2\sum_{s}\epsilon_{s}\left(\mathsf{b}_{s}\mathsf{l}_{s}\,d^{+} + d\,\mathsf{b}_{s}^{+}\mathsf{l}_{s}\right) \\ - \sum_{s}\epsilon_{s}^{-1/2}\left(\mathsf{b}_{s}\sigma_{s}\,d^{+} + d\,\mathsf{b}_{s}^{+}\sigma_{s}\right)\Bigg). \end{aligned} \tag{4.64}$$

By comparing these identities along with the expressions for the leading terms of the relevant formal expansions,

$$\left(\sum_{s}\epsilon_{s}^{\alpha}\left(\mathsf{b}_{s}\sigma_{s}\,d^{+} + d\,\mathsf{b}_{s}^{+}\sigma_{s}\right)\right)_{-1} = m_{\alpha+1/2}, \tag{4.65}$$

$$\left(\sum_{s}\epsilon_{s}^{\beta}\left(\mathsf{b}_{s}\mathsf{l}_{s}\,d^{+} + d\,\mathsf{b}_{s}^{+}\mathsf{l}_{s}\right)\right)_{-1} = 3\sum_{s}\epsilon_{s}^{\beta+1/2}\mathsf{l}_{s}\sigma_{s} - m_{\beta}\boldsymbol{\Lambda}'\cdot\boldsymbol{\sigma}, \tag{4.66}$$

to Eqs. (4.53) and (4.58), the approximate analytic results,

$$U\,h_{osc}\,U^{+} \simeq h_{osc} + \frac{m_{2}}{m_{1}}, \tag{4.67}$$

$$U\,h'_{osc}\,U^{+} \simeq h'_{osc} + \frac{3}{m_{1}}, \tag{4.68}$$

$$U\,\boldsymbol{\Lambda}\cdot\boldsymbol{\sigma}\,U^{+} \simeq -\boldsymbol{\Lambda}\cdot\boldsymbol{\sigma} - m_{1} + \frac{m_{2}}{m_{1}}, \tag{4.69}$$

$$U\,\mathsf{l}\cdot\boldsymbol{\sigma}\,U^{+} \simeq -\mathsf{l}\cdot\boldsymbol{\sigma} + \frac{3}{m_{1}}\sum_{s}\epsilon_{s}\mathsf{l}_{s}\sigma_{s} - \frac{m_{1/2}}{m_{1}}\boldsymbol{\Lambda}'\cdot\boldsymbol{\sigma} + 1 - \frac{m_{1/2}^{2}}{m_{1}}, \tag{4.70}$$

$$U\,\Lambda'\cdot\sigma\,U^{+} \simeq -2\Lambda'\cdot\sigma + \frac{3}{m_1}\sum_s \epsilon_s^{3/2} \mathsf{I}_s \sigma_s - \frac{6}{m_1}, \tag{4.71}$$

for the operator transforms under discussion can be obtained.

In accordance with the Cayley—Hamilton theorem for the deformation matrix $\epsilon = \mathrm{diag}(\epsilon_x, \epsilon_y, \epsilon_z)$, any analytical function of ϵ is a linear combination of three linearly independent functions of ϵ with the coefficients which are invariants of the same matrix (see Appendix C). In particular, it is convenient to make use of the l^0 operator, whose components are defined via

$$l^0{}_r = \frac{1}{2}\sum_{st}(e_{rst})^2 \left(\sqrt{\frac{\epsilon_s}{\epsilon_t}} + \sqrt{\frac{\epsilon_t}{\epsilon_s}}\right) \mathsf{I}_r, \tag{4.72}$$

or, equivalently,

$$l^0{}_r = \frac{1}{2}\sqrt{\epsilon_r}(m_1 - \epsilon_r)\mathsf{I}_r, \tag{4.73}$$

instead of the I vector of the anisotropic orbital momentum. Indeed, by applying Eq. (C.11) from Appendix C, one can derive approximate equations

$$U\,\mathsf{l}^0\cdot\sigma\,U^{+} \simeq -\mathsf{l}^0\cdot\sigma + \frac{m_{-1}}{2m_1}\Lambda'\cdot\sigma - \frac{3}{2m_1}\Lambda\cdot\sigma - 2, \tag{4.74}$$

$$U\,\Lambda'\cdot\sigma\,U^{+} \simeq \Lambda'\cdot\sigma - \frac{6}{m_1}\mathsf{l}^0\cdot\sigma - \frac{6}{m_1} \tag{4.75}$$

to be utilized in place of Eqs. (4.70) and (4.71). Thus, the transforms of the $\mathsf{l}^0\cdot\sigma$ and $\Lambda'\cdot\sigma$ operators can be approximated by the linear combinations of the same operators and $\Lambda\cdot\sigma$ with simple deformation-dependent coefficients.

Note that the operators h_{osc}, n, h'_{osc}, $\Lambda\cdot\sigma$, $\mathsf{l}^0\cdot\sigma$, and $\Lambda'\cdot\sigma$, that comprise a linearly closed set with respect to the pseudospin transformation (4.30), (4.33), can be constructed in a straightforward manner from four structural blocks d, d^{+}, d' and

d'^{+}, where

$$d' = \sum_s \frac{1}{\sqrt{\epsilon_s}} \mathsf{b}_s \sigma_s \tag{4.76}$$

is (with the accuracy of a constant factor) the quanta-decreasing part of the $\mathbf{r}\cdot\boldsymbol{\sigma}$ operator in exactly the same way as d is the part of $\mathbf{p}\cdot\boldsymbol{\sigma}$. All the operators from the closed set are Hermitian quanta-conserving bilinear combinations of the structural blocks and are expressible similarly to Eqs. (4.41) and (4.42):

$$\mathsf{n} + \frac{3}{2} = \frac{1}{4}[d, d'^{+}]_{+} + \text{H.c.}, \tag{4.77}$$

$$h'_{osc} = \frac{1}{2}[d', d'^{+}]_{+}, \tag{4.78}$$

$$\mathbf{l}^0\cdot\boldsymbol{\sigma} + \frac{3}{2} = \frac{1}{4}[d, d'^{+}] + \text{H.c.}, \tag{4.79}$$

$$\boldsymbol{\Lambda}'\cdot\boldsymbol{\sigma} + \frac{m_{-1}}{2} = \frac{1}{2}[d', d'^{+}]. \tag{4.80}$$

Note that since the transforms of the operators of the closed set are linear functions of the operators themselves, and the model Hamiltonian depends only on these operators, the structure of the approximate pseudospin image of the Hamiltonian occurs to be similar to the structure of the Hamiltonian itself. In other words, while the exact transformation of the Hamiltonian cannot be performed in a closed analytical form, the approximate transformation readily yields an effective Hamiltonian of the pseudo Nilsson type. An explicit form of this "pseudo" Hamiltonian will be considered in the next section.

4.2.3 Angular momentum

It follows from the basic definitions (4.3) that the $\mathbf{l}^0$ operator, determined by Eq. (4.72), is the part of the physical orbital momentum operator $\mathbf{l}$ rewritten in the deformed representation which preserves the total number of quanta. As one

can conclude from Eq. (4.79) and the above remarks regarding the d and d' operators, its occurrence in connection to the pseudospin transformation is not accidental and reflects a physically important property of this transformation. Namely, the pseudospin transformation is explicitly defined in the deformed basis but nevertheless is related to the physical angular momentum $\mathbf{j} = \mathbf{l} + \boldsymbol{\sigma}/2$ rather than to the angular momentum $\mathrm{j} = \mathrm{l} + \sigma/2$ in the deformed representation. This fact can be demonstrated by pointing out that the angular momentum can be decomposed into a sum of two operator conjugates,

$$\mathbf{j} = \frac{1}{2}(\boldsymbol{\delta} + \boldsymbol{\delta}^{+}), \tag{4.81}$$

in such a way that

$$[\boldsymbol{\delta}, d] = [\boldsymbol{\delta}^{+}, d^{+}] = 0. \tag{4.82}$$

The Cartesian components of $\boldsymbol{\delta}$ and $\boldsymbol{\delta}^{+}$ are defined via the rule

$$\delta_{\tau} = i \sum_{st} e_{\tau st} \sqrt{\frac{\epsilon_s}{\epsilon_t}} \mathsf{b}_s \left(\mathsf{b}_t + \mathsf{b}_t^{+}\right) + \frac{1}{2}\sigma_{\tau}, \tag{4.83}$$

(and the Hermitian conjugate to this rule) and satisfy the requirement (4.82) by virtue of the standard boson commutation relations (4.4).

Given Eq. (4.82), it is possible to find the approximate transform of the angular momentum vector. Indeed, the commutation rule yields the identity

$$d\,\mathbf{j}\,d^{+} = \frac{1}{2}[\mathbf{j}, dd^{+}]_{+} + \frac{1}{4}[\boldsymbol{\delta} - \boldsymbol{\delta}^{+}, dd^{+}]. \tag{4.84}$$

Since the last term in the r.h.s. of this formula does not contain a contribution proportional to $\mathsf{n} + 3$ (cf. Eqs. (4.53) and (4.57)), the pseudospin transform of $\mathbf{j}$ is

written in the form

$$U\mathbf{j}U^{+} = \mathbf{j} + O(\frac{1}{n+3}), \tag{4.85}$$

in accordance with Eq. (4.58).

The physical angular momentum is therefore conserved by the pseudospin transformation with the better accuracy, the higher the shell number. The degree of conservation of j by the transformation correlates with the degree of its conservation by the Hamiltonian — in line with the rule of conformity between the transformation and the Hamiltonian (see subsection 3.3). In particular, the angular momentum as a whole is conserved in the spherical limit, and its longitudinal component j_z under axial deformation.

Realistic many-particle nuclear Hamiltonians are necessarily rotationally invariant. The rotational symmetry of nuclear systems is reflected by the fact that the helicity transformation (2.3) preserves the angular momenta of individual nucleons:

$$\mathcal{H}\mathbf{j}\mathcal{H} = \mathbf{j} \tag{4.86}$$

while relabeling their orbital and spin momenta. The pseudospin transformation is a modification of the helicity transformation for the needs of the deformed oscillator shell model. Consequently, the rotational invariance is inherited by the pseudospin transformation but only to the extent this can occur under the symmetry breaking due to the deformed mean-field approximation and the condition of a decrease in shell number.

4.2.4 Accuracy of approximate transformation

The analytic approximate transforms were derived so far by means of the general procedure based on the formal expansion in inverse powers of $n+3$. However, for the particular case of Eqs. (4.67) and (4.69) an alternative procedure can be considered which gives some insight into the essence of the approximations as well as an *a priori* estimate of their accuracy.

Indeed, let us suggest that the last term in the r.h.s. of Eq. (4.52) can be approximated by a linear function of $d = \sum_s \epsilon_s^{1/2} b_s \sigma_s$, *i.e.*

$$\sum_s \epsilon_s^{3/2} b_s \sigma_s \simeq \eta\, d, \tag{4.87}$$

where the coefficient η depends only on the deformation. Under this assumption Eq. (4.52) can be applied iteratively to yield

$$d\, f(d\, d^+) \simeq f(d^+ d + 2\eta)\, d. \tag{4.88}$$

The latter formula leads to the approximate transform

$$U\, d\, d^+\, U^+ \simeq d^+ d + 2\eta. \tag{4.89}$$

To determine the optimal value of η as a function of deformation, minimize the norm of the difference operator

$$D(\eta) = \sum_s \left(\epsilon_s^{3/2} - \eta\, \epsilon_s^{1/2}\right) b_s \sigma_s \tag{4.90}$$

with respect to η, where the norm of an operator $\hat{x}$ within the shell number n is defined via its Hermitian "root mean square" value:

$$\|\hat{x}\|_{\mathsf{n}} = \sqrt{\langle xx^{+}\rangle_{\mathsf{n}}}. \tag{4.91}$$

For the case under discussion, one has

$$\|D(\eta)\|_{\mathsf{n}}^{2} = \frac{1}{3}(m_3 - 2m_2\eta + m_1\eta^2)(\mathsf{n}+3). \tag{4.92}$$

The optimal value $\eta = m_2/m_1$, which minimizes the r.h.s. of Eq. (4.92), must be substituted in the r.h.s. of Eq. (4.89). The resulting equation together with Eq. (4.49) is equivalent to Eqs. (4.67) and (4.69).

The minimal value of the squared norm,

$$\left\|D\left(\frac{m_2}{m_1}\right)\right\|_{\mathsf{n}}^{2} = \frac{1}{3}\left(m_3 - \frac{{m_2}^2}{m_1}\right)(\mathsf{n}+3), \tag{4.93}$$

is a deformation-dependent quantitative estimate of the validity of the approximation delivered by Eq. (4.87) or, which is the same, by Eqs. (4.67) and (4.69). Unfortunately, similar estimates are not available for approximations (4.68), (4.74) and (4.75). However, they are of minor importance for the transform of Hamiltonian (4.1) because of the small value of $k\mu$, the coefficient of the u_{ll} term. By evaluating the r.h.s. of the formula (4.93) at various deformations and fixed n, it is easy to find out that the approximation under discussion is more precise for prolate-like shapes than for oblate-like ones, and the accuracy decreases with the deformation increasing.

Thus, the squared norm of the difference operator provides a simple *ad hoc* estimate for the accuracy of the basic approximation used for transforming the model Hamiltonian. Essentially, it measures the accuracy of the approximation to the transformed Nilsson-type Hamiltonian since the exact transform cannot be written in a simple analytical form. As demonstrated below, this measure is also convenient for the analysis of the relative accuracy of different models.

It is interesting that the Nilsson-type model, developed in this chapter, can be viewed as a further refinement over the triaxial model with the deformed orbital momentum that was intended to correctly reproduce the structure of the basis states in both the spherical and cylindrical limits and to extend this interpolation to arbitrary triaxial shapes [39]. The general structure of the Hamiltonian of the latter model is similar to Eq. (4.1) while the harmonic oscillator term (4.2) is standard, and the spin-orbit and orbit-orbit interactions are defined by analogy to Eq. (4.6) but with the spherical-representation operators l and n replaced by the deformed-representation operators l and n, respectively:

$$v_{ls} = \mathsf{l}\cdot\boldsymbol{\sigma},\ v_{ll} = \mathsf{l}^2 - \langle \mathsf{l}^2 \rangle_{\mathsf{n}}. \tag{4.94}$$

(To be precise, the $\langle \mathsf{l}^2 \rangle_{\mathsf{n}}$ correction was absent in the original Hamiltonian but it must be inserted for preserving the shell-model structure.)

Because of the analogy between the structure of the u_{ls} and u_{ll} terms in this model and in the spherical limit it is natural to construct the appropriate pseudospin transformation merely as the deformed analog of the transformation (4.9–4.11), *i.e.*

by rewriting the structural block in the form

$$d_{def} = \sum_s \mathsf{b}_s \sigma_s. \tag{4.95}$$

It is important to note that d_{def} commutes with the deformed angular momentum operator j. As a result, the corresponding transformation operator U_{def} leaves j invariant,

$$U_{def}\, \mathsf{j}\, U_{def}^{+} = \mathsf{j}, \tag{4.96}$$

in contrast to the pseudospin transformation U which approximately preserves the physical angular momentum j (see Eq. (4.85). Therefore, although the eigenstates in both the models coincide in the spherical and cylindrical limits, and the energy spectra in the spherical limit are the same, the pseudospin transformation-based model seems to be more adequate in general. However, within the experimentally attainable domain of deformations it is hard to expect a serious difference in predictions of the two models.

It follows from Eqs. (4.15), (4.16) and (4.17) that the transforms

$$U_{def}\, \mathsf{n}\, U_{def}^{+} = \mathsf{n} + 1, \tag{4.97}$$

$$U_{def}\, \mathsf{l}\cdot\boldsymbol{\sigma}\, U_{def}^{+} = -\mathsf{l}\cdot\boldsymbol{\sigma} - 2, \tag{4.98}$$

$$U_{def}\, \mathsf{l}^2\, U_{def}^{+} = \mathsf{l}^2 + 2\mathsf{l}\cdot\boldsymbol{\sigma} + 2, \tag{4.99}$$

are exact. However, the h_{osc} term can be transformed only approximately, by using the techniques similar to those developed in the previous subsections. By combining

those techniques with identities

$$d_{def}\, h_{osc}\, d_{def}^{+} = \frac{1}{2}\left([h_{osc},\, d_{def}\, d_{def}^{+}]_{+} + \sum_{s} \epsilon_s \left(\mathsf{b}_s \sigma_s\, d_{def}^{+} + d_{def}\, \mathsf{b}_s^{+} \sigma_s\right)\right), \quad (4.100)$$

$$d_{def}\, d_{def}^{+} = \mathsf{n} + 3 + \mathsf{l}\cdot\sigma, \quad (4.101)$$

it is possible to obtain the approximate analytic expression

$$U\, h_{osc}\, U^{+} \simeq h_{osc} + \frac{m_1}{3}. \quad (4.102)$$

Therefore, the approximate image of the Hamiltonian can be written in the pseudo Nilsson form

$$U_{def}\, h\, U_{def}^{+} \simeq h_{osc} + \frac{m_1}{3} - k((2\mu - 1)v_{ls} + \mu v_{ll}) + k(1-\mu) + k\mu(\mathsf{n}+2), \quad (4.103)$$

which is reminiscent of Eq. (4.18).

The approximate transform (4.102) and, consequently, (4.103), is alternatively derivable by means of the approximation rule,

$$\sum_{s} \epsilon_s \mathsf{b}_s \sigma_s \simeq \eta'\, d_{def}, \quad (4.104)$$

which is the analog of (4.87). By pursuing the analogy further, measure the accuracy of this rule by assessing the minimal value of the squared norm of the corresponding difference operator. Similarly to Eq. (4.93), the numerical value of this measure is given by

$$\left\| D_{def}\left(\frac{m_1}{3}\right)\right\|_{\mathsf{n}}^{2} = \frac{1}{3}\left(m_2 - \frac{{m_1}^2}{3}\right)(\mathsf{n}+3). \quad (4.105)$$

The ratio of the r.h.s.' of the former and the latter equations provides a quantitative shell-independent estimate for the accuracy of the approximation rule (4.87) (within the model developed in this chapter) relative to the accuracy of rule (4.104) (within the model with the deformed orbital momentum):

$$R = \frac{\|D(\frac{m_2}{m_1})\|_n^2}{\|D_{def}(\frac{m_1}{3})\|_n^2} = \frac{m_3 - \frac{m_2{}^2}{m_1}}{m_2 - \frac{m_1{}^2}{3}}, \tag{4.106}$$

where $m_k, k = 1, 2, 3$ are functions only on the dimensionless frequencies ϵ_z, ϵ_x and ϵ_y (see Eq. (4.44)).

It is convenient to parametrize the frequencies in (β, γ)-terms [27]:

$$\epsilon_z = v(1 - 2\beta\cos\gamma), \tag{4.107}$$

$$\epsilon_x = v(1 - 2\beta\cos(\gamma - 2\pi/3)), \tag{4.108}$$

$$\epsilon_y = v(1 - 2\beta\cos(\gamma + 2\pi/3)), \tag{4.109}$$

where

$$v = (1 - 3\beta^2 - 2\beta^3\cos 3\gamma)^{-1/3} \tag{4.110}$$

is the factor providing the volume conservation. This parametrization maps the region of possible deformations (under the $\epsilon_z \leq \epsilon_x \leq \epsilon_y$ constraint that makes the choice unique) into the triangle in the (β, γ)-plane with the boundaries $0 \leq \gamma \leq \pi/3$ and $0 \leq \beta\cos\gamma \leq 1/2$. For small deformations,

$$\beta \simeq \sqrt{\frac{5}{16\pi}}\beta_B, \quad \gamma \simeq \gamma_B,$$

where β_B and γ_B are the conventional Bohr parameters [7].

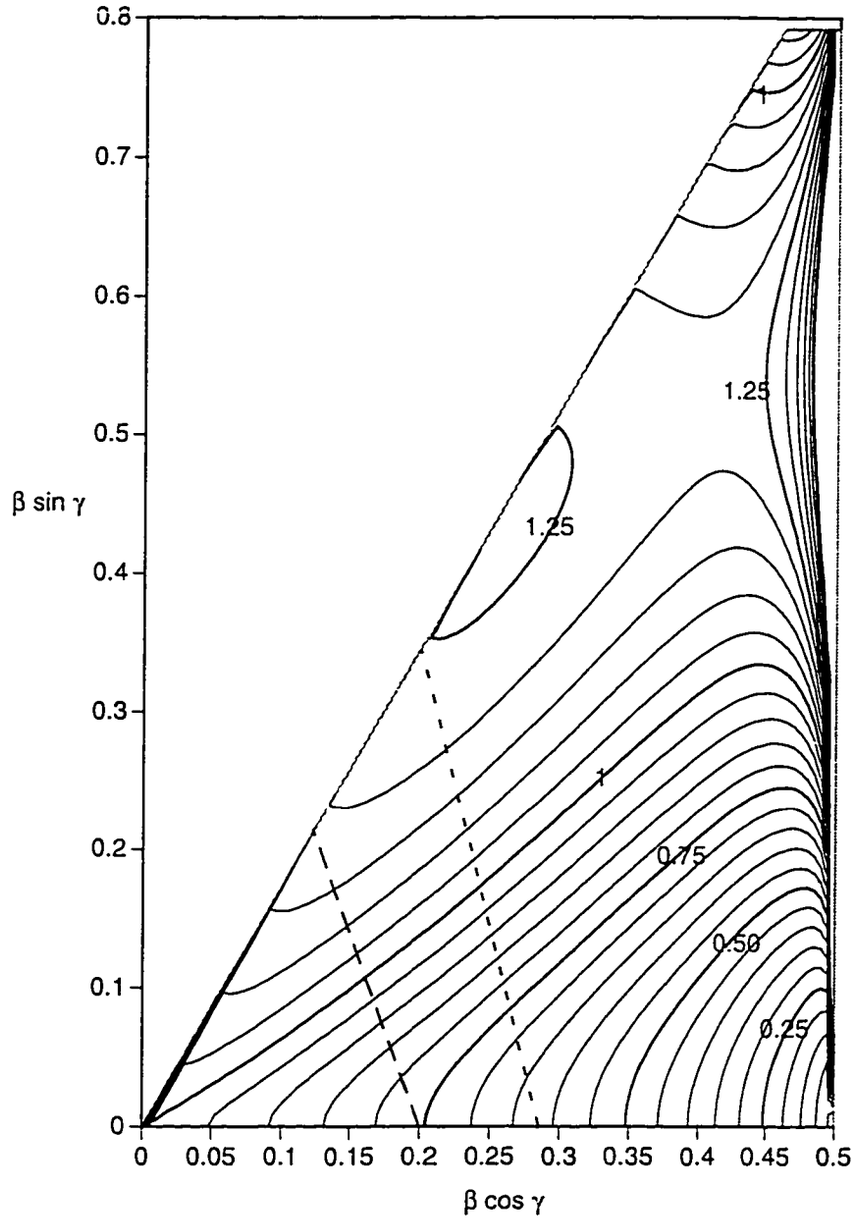

Figure 4.1: Deformation dependence of the ratio $\frac{m_3-\frac{m_2^2}{m_1}}{m_2-\frac{m_1^2}{3}}$. Dashed lines mark the areas of superdeformation and hyperdeformation (ratio of the maximal frequency to the minimal one is 2 and 3, respectively).

By means of the above parametrization, the ratio (4.106) can be rewritten in the form

$$R = \frac{1 - \beta \cos 3\gamma - 2\beta^2}{(1 - 3\beta^2 - 2\beta^3 \cos 3\gamma)^{1/3}}. \tag{4.111}$$

The corresponding plot is displayed in Fig. (4.1). It clearly demonstrates that in the region $0 \leq \gamma < \pi/6$ (which contains the majority of strongly deformed nuclei), approximation (4.87) is preferable, falling closer to the cylindrical limit. For $\pi/6 < \gamma \leq \pi/3$ the rule (4.104) is slightly more satisfactory, though in the asymptotic oblate limit the former approximation is again more accurate. The only domain where the approximation (4.87) definitely loses the competition is the experimentally unreachable region of asymptotically high nonaxial deformations. For the realistic nuclear deformations, both the approximations have comparable precision ($0.7 < R < 1.3$).

4.2.5 Limiting cases

The approximate analytic transforms (4.67), (4.68), (4.69), (4.74) and (4.75) were derived exclusively by means of the techniques based on formal expansions in inverse powers of $n + 3$. Therefore, it is expedient to analyze their functional forms in the familiar limiting cases and compare to the available exact results.

In the spherical limit, when

$$h_{osc}, h'_{osc} \to n + \frac{3}{2}; \quad \Lambda, \mathrm{l}^0, \Lambda' \to \mathrm{l}, \tag{4.112}$$

and $m_k = 3$ for any k, Eqs. (4.67) and (4.68) are equivalent to (4.15), and Eqs. (4.69), (4.74) and (4.75) to (4.16). The transform (4.17) of the l^2 operator is then derivable by means of Eq. (4.39). All the results are necessarily exact.

Considering the axial case ($\epsilon_x = \epsilon_y = \epsilon, \epsilon_z = \epsilon^{-2}$) serves a double purpose. First, the presence of degeneracy associated with the axial symmetry provides a strong consistency check for two triads of equations, namely, (4.67), (4.51), (4.68) and (4.69), (4.74), (4.75). Indeed, equations within these triads must become linearly dependent and mutually noncontradictory — otherwise the approach would be flawed. Second, the cylindrical and asymptotic oblate limits are easily extractable from the axial case formulae.

A direct check shows that both the triads meet the consistency test. Then the application of the definitions (4.2), (4.43), and (4.47) corroborates that the two triads in the axial case are equivalent to two dyads, for the longitudinal and transverse components of the number of quanta and spin-orbit splitting operators:

$$U\,\mathsf{n}_z\,U^+ \simeq \mathsf{n}_z + \frac{1}{2\epsilon^3+1}, \tag{4.113}$$

$$U\,\mathsf{n}_\rho\,U^+ \simeq \mathsf{n}_\rho + \frac{2\epsilon^3}{2\epsilon^3+1}, \tag{4.114}$$

$$U\,\mathsf{l}_z\sigma_z\,U^+ \simeq -\frac{(2\epsilon^3-1)\mathsf{l}_z\sigma_z + \epsilon^{3/2}\mathsf{l}_\rho\sigma_\rho + 2\epsilon^3}{2\epsilon^3+1}, \tag{4.115}$$

$$U\,\mathsf{l}_\rho\sigma_\rho\,U^+ \simeq -\frac{(\epsilon^3+1)\mathsf{l}_\rho\sigma_\rho + 2\epsilon^{3/2}\mathsf{l}_z\sigma_z + 4\epsilon^{3/2}}{2\epsilon^3+1}, \tag{4.116}$$

where $\mathsf{l}_\rho\sigma_\rho$ stands for $\mathsf{l}_x\sigma_x + \mathsf{l}_y\sigma_y$.

In the cylindrical limit ($\epsilon \gg 1$), approximate transforms (4.113), (4.114), and (4.115) become exact and coincide with (4.26), (4.25) and (4.27), respectively. Since Eq. (4.28) is just the second power of Eq. (4.27), it follows from (4.115) as well. However, the image

$$U_\infty\,\mathsf{l}_\rho\sigma_\rho\,U_\infty^+ \simeq -\frac{1}{2}\mathsf{l}_\rho\sigma_\rho \tag{4.117}$$

remains approximate because the exact image cannot be expressed through the components of the spin-orbit splitting operator only.

The connection among the approximate and exact images in the asymptotic oblate limit ($\epsilon \to 0$) is quite similar. The transformation operator in this limit has a form

$$U_{-\infty} = \frac{1}{\sqrt{n_z + 1}} b_z \sigma_z, \tag{4.118}$$

and equations

$$U_{-\infty}\, n_\rho\, U^+_{-\infty} = n_\rho, \tag{4.119}$$

$$U_{-\infty}\, n_z\, U^+_{-\infty} = n_z + 1, \tag{4.120}$$

$$U_{-\infty}\, l_z\, \sigma_z U^+_{-\infty} = l_z \sigma_z, \tag{4.121}$$

$$U_{-\infty}\, l_z^2\, U^+_{-\infty} = l_z^2, \tag{4.122}$$

which follow from (4.113), (4.114) and (4.115), are exact. The image

$$U_{-\infty}\, l_\rho \sigma_\rho\, U^+_{-\infty} \simeq -l_\rho \sigma_\rho \tag{4.123}$$

is approximate again; the exact image is not expressible through the transverse components of the orbital and spin momenta only but reaches the limit (4.123) at $n_z \gg 1$.

A general observation, which ensues from the analysis of this subsection and Eq. (4.85), is that the procedure of approximate pseudospin transformation provides exact results for the integrals of motion and reasonable approximate expressions for the rest of the operators.

4.3 HAMILTONIAN IN PSEUDOSPIN REPRESENTATION

Since the set of necessary operator transforms has been obtained, it is now possible to proceed with the transformation of the model Hamiltonian. This is a crucial point for the approach because all the previous results are to be checked for conformity.

The Nilsson-type Hamiltonian, associated with the deformed pseudospin transformation (4.30), (4.33), is determined by Eqs. (4.1), (4.2), (4.42–4.48) and includes two deformation-dependent positive parameters, ζ and ξ, which reach the value of 1 in the spherical limit. These dependences are to be chosen in order to satisfy the third general consistency requirement from subsection 4.2.3, namely the physical equivalence of the spherical and deformed pseudo representations.

Consider the model Hamiltonian at $\mu = 0.5$,

$$h\,|_{\mu=0.5} = h_{osc} - k\left(\zeta \boldsymbol{\Lambda}\cdot\boldsymbol{\sigma} + \frac{\xi}{2}\left(\boldsymbol{\Lambda}^2 - \langle\boldsymbol{\Lambda}^2\rangle_n\right)\right). \tag{4.124}$$

Since this value of μ corresponds to the "exact" pseudospin limit, the spin-orbit splitting term in the transformed Hamiltonian should vanish or, at least, acquire the minimal magnitude possible.

By applying the approximate transformation rule,

$$U\boldsymbol{\Lambda}^2U^+ \simeq \boldsymbol{\Lambda}^2 + \frac{2}{m_1}(2m_{-1}\boldsymbol{\Lambda}\cdot\boldsymbol{\sigma} - 3\mathrm{l}^0\cdot\boldsymbol{\sigma}) + 2\frac{m_1 + 2m_{-2}}{{m_1}^2}, \tag{4.125}$$

which follows from Eqs. (4.48), (4.69), and (4.75), as well as from Eq. (4.67) and Eq. (4.69) itself, it is possible to obtain the following spin-orbit term in the approx-

imate image of the Hamiltonian (4.124):

$$\frac{3}{m_1}k\xi\left[\frac{1}{3}\left(2m_{-1}-m_1\frac{\zeta}{\xi}\right)\boldsymbol{\Lambda}\cdot\boldsymbol{\sigma}-\mathrm{l}^0\cdot\boldsymbol{\sigma}\right]. \tag{4.126}$$

This term disappears in the spherical limit and may vanish in the cylindrical one (provided the ratio ξ/m_1 decreases at a proper rate) but it cannot vanish at arbitrary deformation. Instead, the ratio ξ/ζ must be chosen to minimize the norm of the operator in the square brackets.

Define the deformation-dependent coefficient

$$\breve{\zeta}=\frac{1}{3}\left(2m_{-1}-m_1\frac{\zeta}{\xi}\right) \tag{4.127}$$

and minimize the squared norm of the difference operator

$$C(\breve{\zeta})=\mathrm{l}^0\cdot\boldsymbol{\sigma}-\breve{\zeta}\boldsymbol{\Lambda}\cdot\boldsymbol{\sigma} \tag{4.128}$$

with respect to $\breve{\zeta}$. (The norm for a given operator is defined in Eq. (4.91).) By using Eqs. (4.42), (4.72) and the formulae $\langle \mathrm{l}_s\sigma_s\rangle_{\mathsf{n}}=0$ and $\langle \mathrm{l}_s{}^2\rangle_{\mathsf{n}}=\mathsf{n}(\mathsf{n}+3)/6$, $s=z,x,y$, it can be shown that for $C(\breve{\zeta})$ within the shell number n the squared norm equals to

$$\|C(\breve{\zeta})\|^2_{\mathsf{n}}=\left(\frac{1}{4}(3+m_1m_{-1})-2m_1\breve{\zeta}+m_{-1}\breve{\zeta}^2\right)\frac{\mathsf{n}(\mathsf{n}+3)}{6}, \tag{4.129}$$

and its minimum is reached at

$$\breve{\zeta}_{opt}=\frac{m_1}{m_{-1}}. \tag{4.130}$$

In accordance with Eq. (4.127) and the identity (C.12) from Appendix C, this result yields the relation

$$\xi = \frac{m_1 m_{-1}}{m_1 + 2m_{-2}}\zeta. \tag{4.131}$$

Now integrate the results of this section into the approximate transformation rule

$$\begin{aligned} U(u_{ls} + \mu u_{ll})U^+ \simeq\ & (2\mu - 1)u_{ls} + 6\mu\frac{\xi}{m_1}\left(\frac{\breve{\zeta}_{opt}}{\zeta}u_{ls} - \mathbf{l}^0\cdot\boldsymbol{\sigma}\right) + \mu u_{ll} \\ & -2\frac{\zeta}{\breve{\zeta}_{opt}}(1-\mu) - \frac{1}{3}\mu m_{-1}\xi(\mathsf{n}+2). \end{aligned} \tag{4.132}$$

The analysis of this rule gives a unique prescription for ζ, namely

$$\zeta = \breve{\zeta}_{opt} = \frac{m_1}{m_{-1}}. \tag{4.133}$$

Indeed, according to this choice and Eq. (4.128), the u_{ls} operator becomes the best approximation to the $\mathbf{l}\cdot\boldsymbol{\sigma}$ operator given the constraints of the conservation of quanta in the deformed representation and the conformity with the pseudospin transformation (4.30), (4.33). This in turn implies that the Hamiltonian thus obtained combines the advantages of the conventional Nilsson scheme (physical vs deformed orbital momentum) with the general requirements from subsection 4.2.3.

Finally, the transformed Hamiltonian acquires the following form:

$$\begin{aligned} U\,h\,U^+ \simeq\ & h_{osc} + \frac{m_2}{m_1} - k((2\mu-1)u_{ls} + \mu u_{ll}) \\ & + k\mu\,\delta u_{ls} + 2k(1-\mu) + \frac{1}{3}k\mu m_{-1}\xi(\mathsf{n}+2), \end{aligned} \tag{4.134}$$

where

$$u_{ls} = \frac{m_1}{m_{-1}} \boldsymbol{\Lambda} \cdot \boldsymbol{\sigma}, \tag{4.135}$$

$$u_{ll} = \xi(\boldsymbol{\Lambda}^2 - \langle \boldsymbol{\Lambda}^2 \rangle_n), \tag{4.136}$$

$$\delta u_{ls} = 6\frac{\xi}{m_1}\left(u_{ls} - \mathbf{l}^0 \cdot \boldsymbol{\sigma}\right), \tag{4.137}$$

$$\xi = \frac{{m_1}^2}{m_1 + 2m_{-2}}. \tag{4.138}$$

Given this formula, one can check the fulfillment of the equivalence requirement between the spherical and deformed representations and then numerically test isospectral character of the original and transformed Hamiltonians in the regular sector.

The equivalence requirement states that the operations of altering the deformation of and transforming the Hamiltonian are essentially interchangeable. In other words, the image (4.134) of the Hamiltonian at a given deformation can also be obtained by putting the deformation into the pseudo Nilsson Hamiltonian of the spherical limit (4.18). This requirement ensures that the conventional way of doing shell-model calculations in the pseudospin-adapted basis [10, 42, 46], which utilizes the diagonalization of the many-particle Hamiltonian in the spherical pseudo representation (and thus generates the deformation dynamically), is physically equivalent to the diagonalization of the Hamiltonian already transformed in the framework of the shell-model basis with a corresponding deformation.

Since the general structure of the model Hamiltonian (4.1) remains the same at any deformations, and the only varying component is the content of the h_{osc}, u_{ls} and u_{ll} terms as a function of oscillator frequencies, the operation of deforming converts

the transformed spherical Hamiltonian (4.18) into the "pseudo" form

$$U\,h\,U^{+} = h_{osc} + \frac{m_2}{m_1} - k\left((2\mu-1)u_{l_s} + \mu u_{ll}\right) + 2k(1-\mu) + k\mu(\mathsf{n}+2). \qquad (4.139)$$

(Note that the deformation-dependent term $m_2/m_1 - 1$, which only shifts the energy spectrum as a whole, was added to the r.h.s. of this equation for conformity with the rule (4.67).)

Ideally, the r.h.s. of Eqs. (4.134) and (4.139) should coincide. In fact, they are essentially the same. A slight distinction exists because of the presence of δu_{l_s} in the former equation and the difference in the coefficient at $k\mu(\mathsf{n}+2)$ (which is $m_{-1}\xi/3$ in (4.134) vs 1 in (4.139)). However, an estimate by means of the operator norms demonstrates that the distinction is not significant, at least within the experimentally attainable domain of deformations. The relevant data are presented graphically in Figs. (4.2) and (4.3). The first of the plots roughly evaluates the magnitude of spin-orbit term in (4.134) relative to the spin-orbit splitting magnitude in (4.139) by means of the ratio $\frac{\|(2\mu-1)u_{l_s}+\mu\delta u_{l_s}\|}{\|(2\mu-1)u_{l_s}\|}$ for neutrons ($\mu = 0.4$). The estimated ratio is about 1.1 for normal rotational bands (lantanides and actinides) and reaches 1.4 in the hyperdeformation region, and decreases with increasing γ, the nonaxiality angle. In the region of asymptotically high deformations the ratio reaches 1 again which means that the δu_{l_s} term vanishes. The deformation dependence of the coefficient $m_{-1}\xi/3$ is plotted in Fig. (4.3). This coefficient is very close to 1 (within 10%) up to the hyperdeformation region, and stays within 30% practically at any imaginable shapes except for the unreachable region of asymptotically strong oblate deformation.

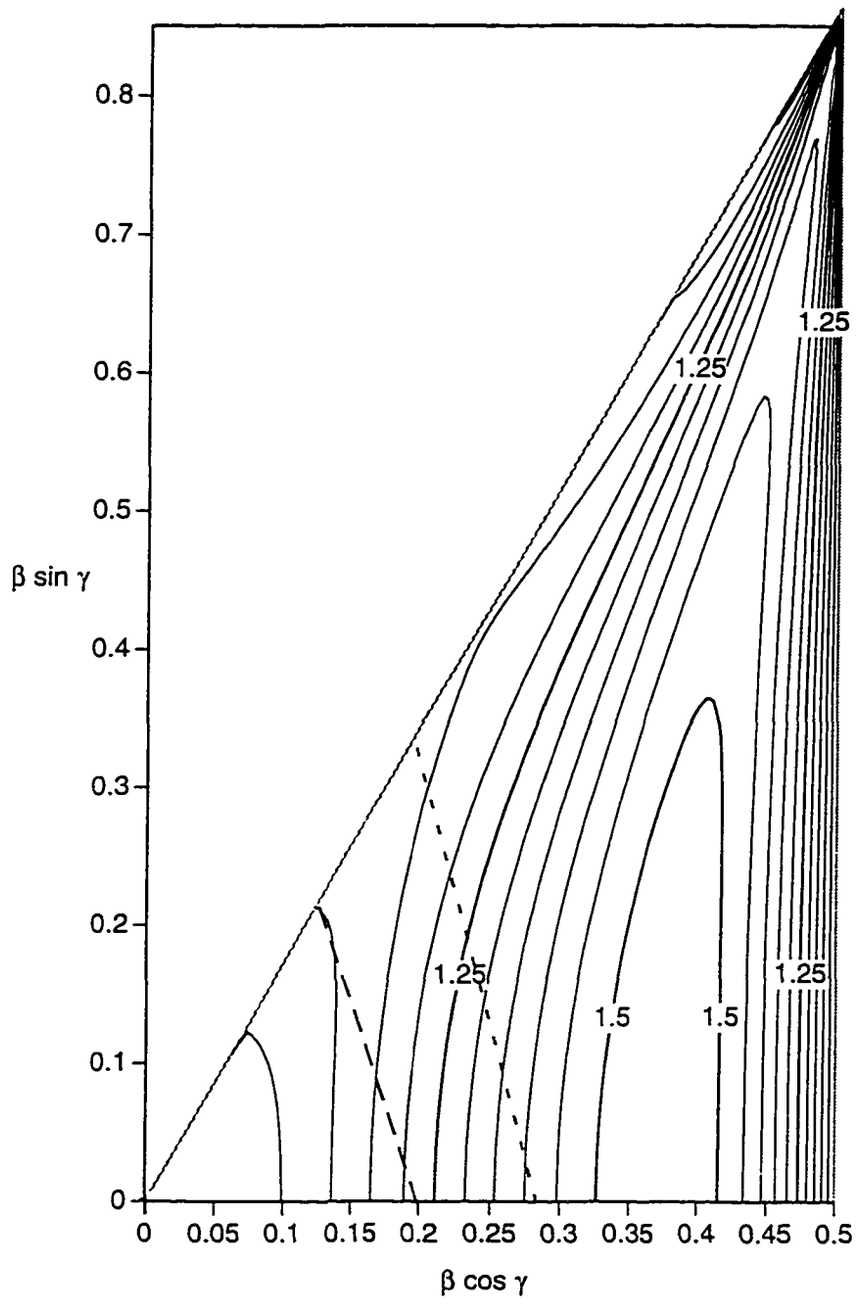

Figure 4.2: Deformation dependence of the ratio $\frac{\|(2\mu-1)u_{l_s}+\mu\delta u_{l_s}\|}{\|(2\mu-1)u_{l_s}\|}$ for neutrons ($\mu = 0.4$). Dashed lines mark the areas of superdeformation and hyperdeformation.

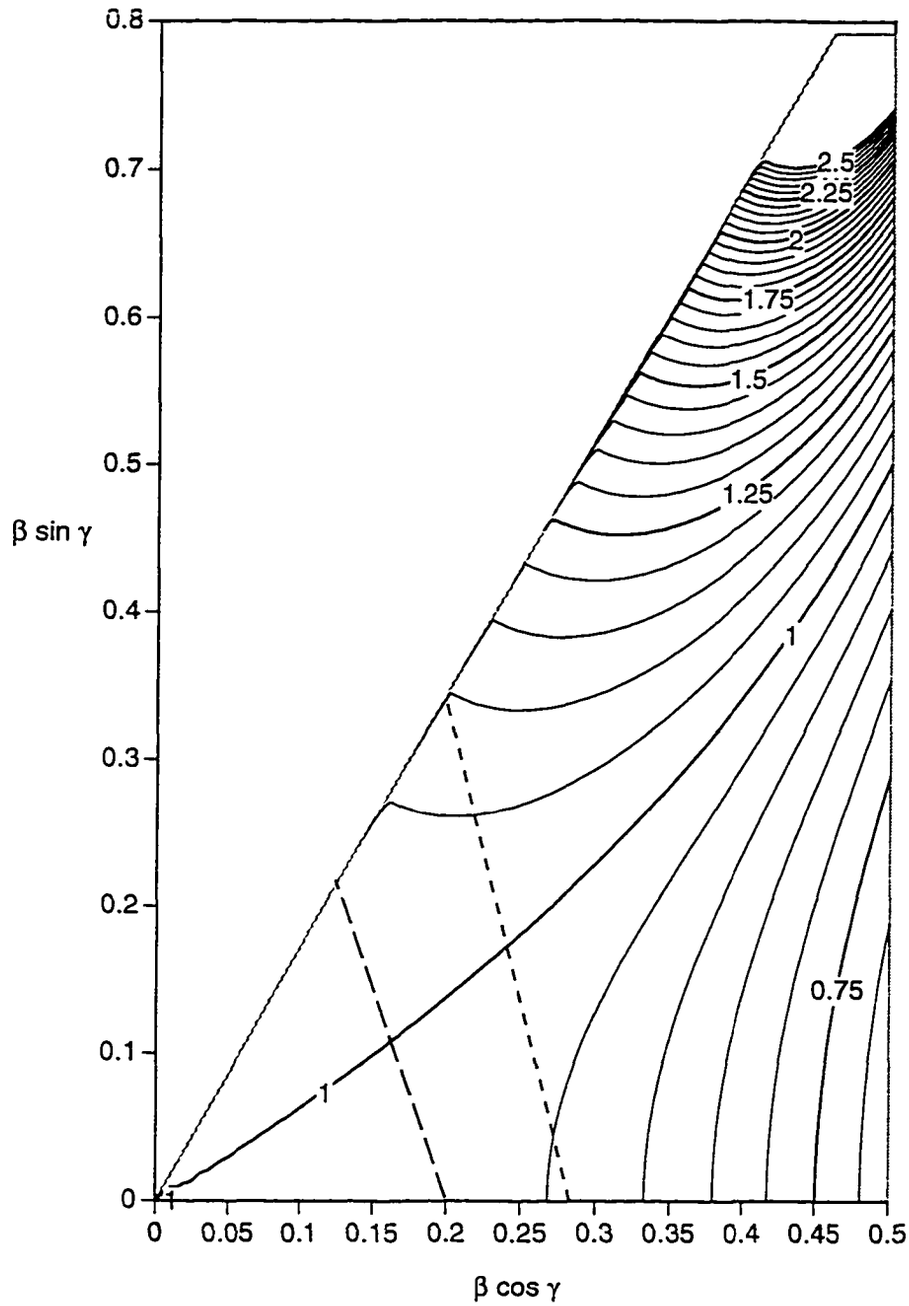

Figure 4.3: Deformation dependence of the coefficient $m_1\xi/3$. Dashed lines mark the areas of superdeformation and hyperdeformation.

Therefore, the *a priori* criteria demonstrate an excellent fulfillment of the equivalence requirement for all attainable deformations and a very reasonable fulfillment over the whole (β, γ)-plane. This means that within the accuracy of the approximate pseudospin transformation the pseudospin dynamical symmetry is valid and can be used reliably in calculations of any deformed heavy nuclei in the spherical, as well as deformed, shell-model basis.

The final test for the entire approach is a numerical calculation. It serves as an integral measure of all the approximations involved and also of the equivalence requirement. The reason for this statement is that in the absence of approximations the spectra of the original and transformed Hamiltonians would be identical in the regular sector of the single-particle space of states. The test calculation has been done with the deformation parameter β spanning the interval up to 0.4 and different values of the nonaxiality angle γ. This deformation domain extends out to the hyperdeformation area and completely covers the region of experimental interest. The employed values of the parameters $k = 0.0637$, $\mu = 0.42$ are characteristic of the lantanides. Since the model Hamiltonian conserves the number of quanta in the deformed representation, the calculation was confined to the neutron shells with $\mathfrak{n} = 5$ and $\tilde{\mathfrak{n}} = 4$ which are relevant for the rare-earth region of the periodical system.

The calculated spectra are plotted in Figs. (4.4), (4.5), (4.6), (4.7), (4.8), and (4.9). First of all, observe that some of the levels of the original Hamiltonian are not reproduced by the Hamiltonians acting in the pseudo representation. In the spherical limit they are known as defector levels and belong to a single j subshell of the given oscillator shell with the maximal possible angular momentum j. This is no longer true in the presence of deformation; however, the separation of the space of states into the regular and singular sectors with respect to the pseudospin

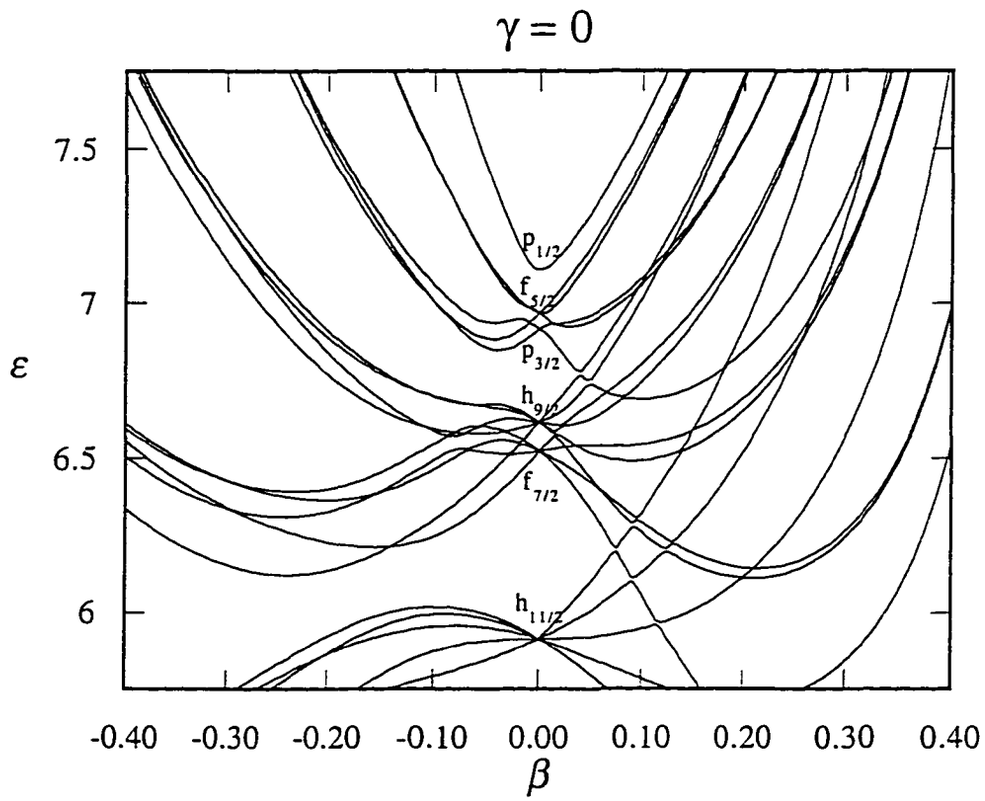

Figure 4.4: Neutron levels of the original Hamiltonian. Axial case. Basis is confined to the deformed oscillator shell $n = 5$. Model parameters: $k = 0.0637, \mu = 0.42$. Energies in units of $\hbar\omega$. Positive values of β correspond to prolate deformation, negative values to oblate deformation. (Use of negative β is based upon the physical equivalence of $(-\beta, \gamma)$ and $(\beta, \pi/3 - \gamma)$ parameter sets.)

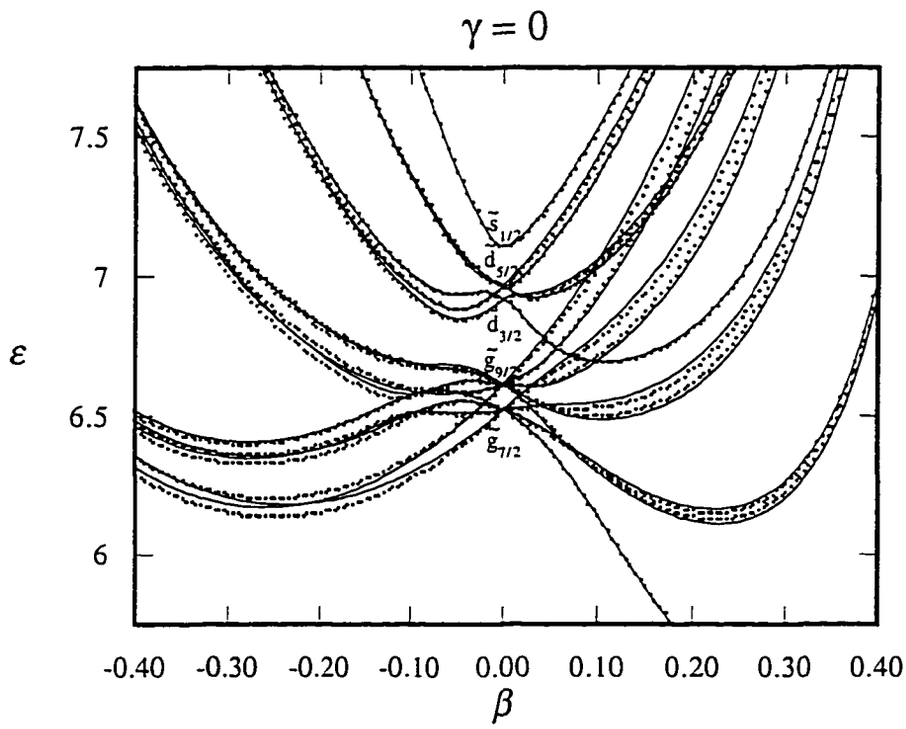

Figure 4.5: Neutron levels of the transformed and "pseudo" Hamiltonians. Axial case. Basis is confined to the deformed oscillator shell $\tilde{n} = 4$. Model parameters: $k = 0.0637, \mu = 0.42$. Energies in units of $\hbar\omega$. Positive values of β correspond to prolate deformation, negative values to oblate deformation. Continuous lines correspond to the transformed Hamiltonian; dotted lines to the "pseudo" version.

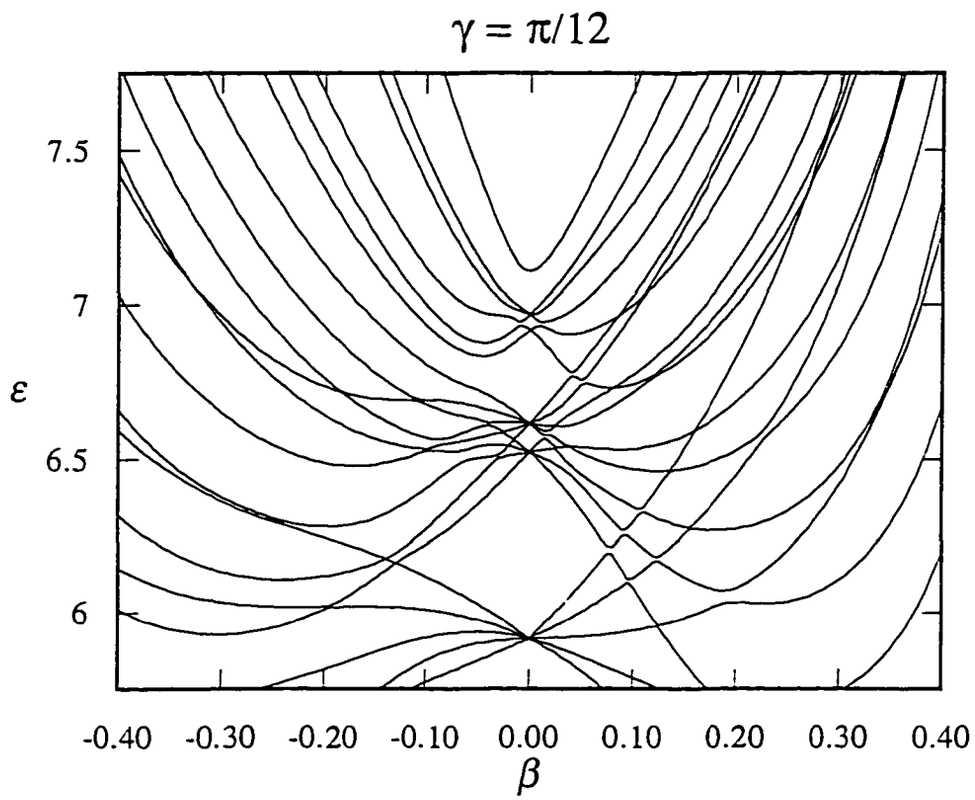

Figure 4.6: Neutron levels of the original Hamiltonian. Intermediate triaxiality. Basis is confined to the deformed oscillator shell $\mathfrak{n} = 5$. Model parameters: $k = 0.0637, \mu = 0.42$. Energies in units of $\hbar\omega$. Positive values of β correspond to $\gamma = \pi/12$, negative values to $\gamma = \pi/4$.

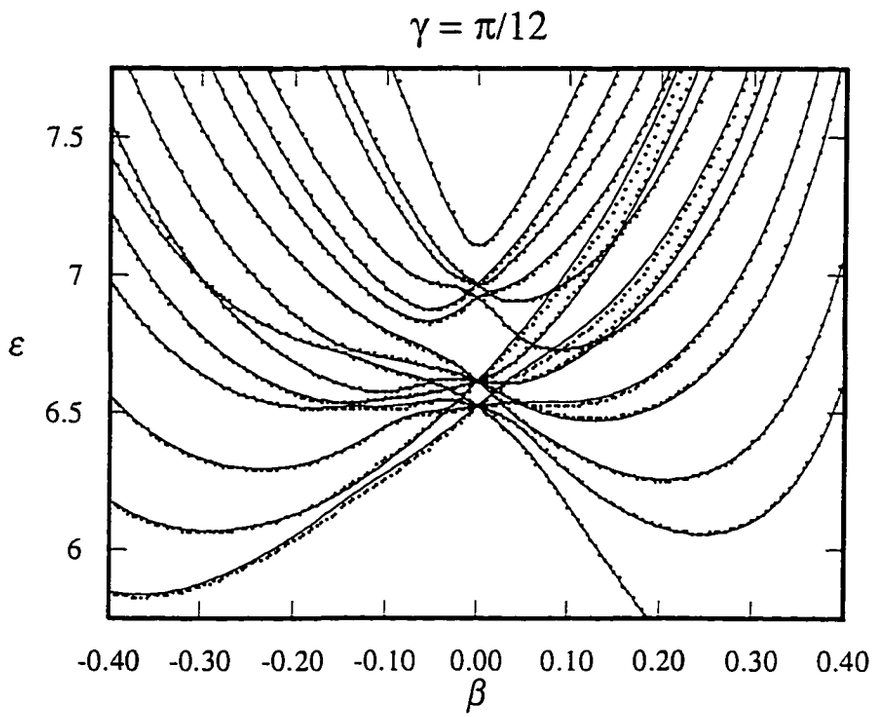

Figure 4.7: Neutron levels of the transformed and "pseudo" Hamiltonians. Intermediate triaxiality. Basis is confined to the deformed oscillator shell $\tilde{n} = 4$. Model parameters: $k = 0.0637, \mu = 0.42$. Energies in units of $\hbar\omega$. Positive values of β correspond to $\gamma = \pi/12$, negative values to $\gamma = \pi/4$.

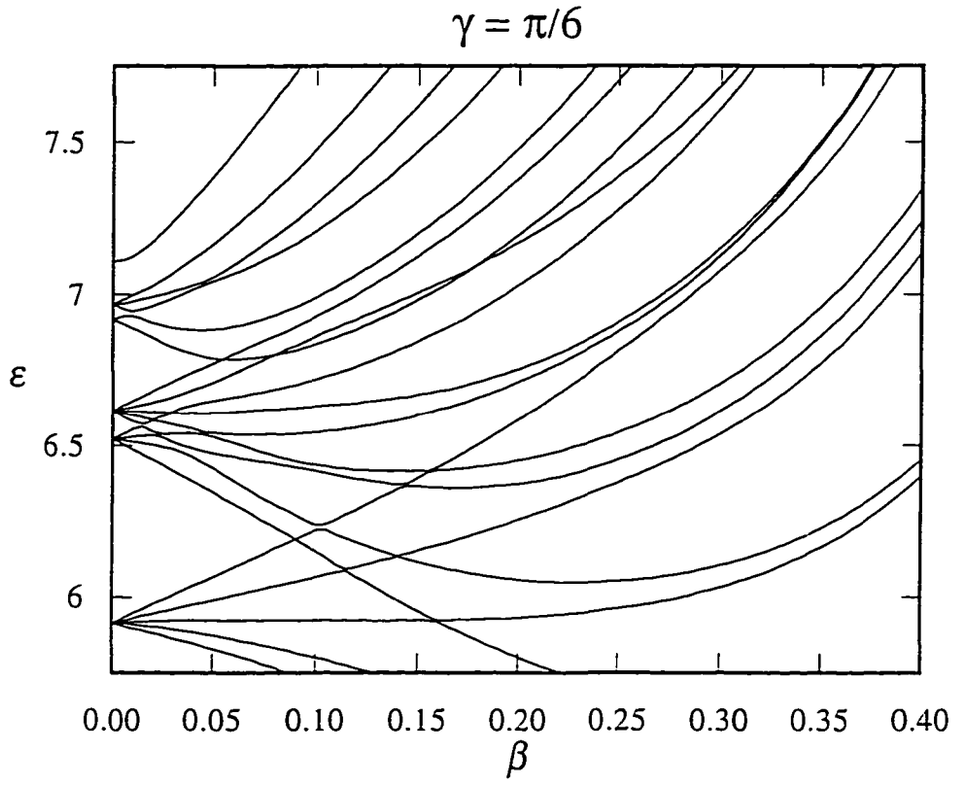

Figure 4.8: Neutron levels of the original Hamiltonian. Maximal triaxiality. Basis is confined to the deformed oscillator shell $n = 5$. Model parameters: $k = 0.0637, \mu = 0.42$. Energies in units of $\hbar\omega$.

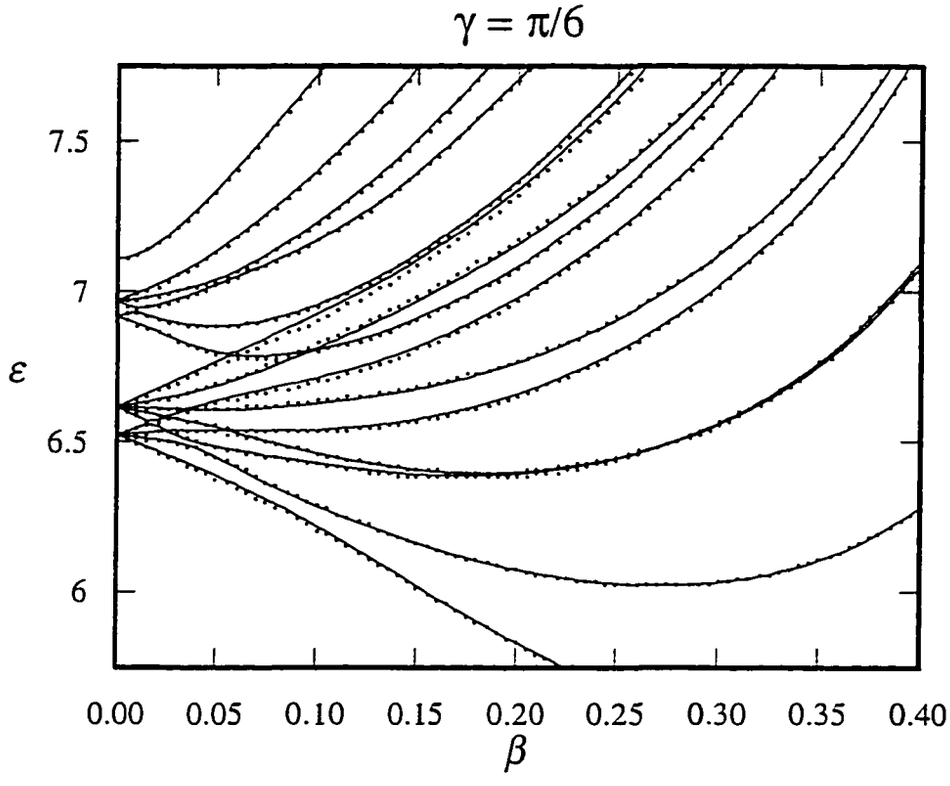

Figure 4.9: Neutron levels of the transformed and "pseudo" Hamiltonians. Maximal triaxiality. Basis is confined to the deformed oscillator shell $\tilde{n} = 4$. Model parameters: $k = 0.0637, \mu = 0.42$. Energies in units of $\hbar\omega$.

transformation depends smoothly on the nuclear shape parameters. As for the regular sector, within this vast deformation domain the corresponding energy levels of the transformed and "pseudo" Hamiltonians are very close and in most cases practically coincide which is a direct validation of the equivalent requirement. The spectra in the pseudo representation fairly closely (with the difference in energy no more that 3–4%, usually much less) follow the "parent" levels of the original Hamiltonian. Note that for the prolate-like shapes ($\gamma \leq \pi/6$) the energy levels tend to be reproduced with better accuracy than for the oblate-like shapes ($\gamma \geq \pi/6$). Such an asymmetry is in line with the *ad hoc* estimates in subsection 4.3.4 and, in particular, with Fig. (4.1). From the practical standpoint, this situation is the most favorable since it reflects the observed correlation between the strong deformation and the prolate shape of nuclei. Consequently, the approximation displays the high accuracy exactly where it is required by nature.

4.4 Concluding remarks

An explicit deformation-dependent form of the pseudospin transformation has been suggested and studied in the framework of the single-particle harmonic oscillator shell model. By construction, the transformation operator is a special quanta-decreasing projection of the momentum helicity transformation which in turn is known to accomplish the relevant relabeling of the spin and orbital momenta in the scope of more realistic mean-field and many-particle approaches to nuclear structure [6]. The connection between the two transformations is displayed in detail. In the limiting cases of both the spherical and strongly prolate nuclear shapes, the deformation-dependent pseudospin transformation reduces to already familiar functional forms [13, 15].

The results of transforming the single-nucleon space of states and the Hamiltonian are found to corroborate and to qualitatively describe the concept of dynamical pseudospin symmetry at any reasonable deformation. The concept offers an effective description of the many-nucleon systems since the basis of single-particle states involves only a subset of the entire set of the relevant states. However, this subset is what primarily contributes to the dynamical generation of deformation for the nucleus as a whole. In the normal representation it coincides with the region of unitarity of the pseudospin transformation and by this reason is determined in a consistent manner. The basic idea of the dynamical pseudospin symmetry concept is that in the pseudospin representation the strength of the spin-orbit splitting term of the single-particle potential is drastically reduced regardless of the degree of deformation and nonaxiality of the nuclear shape. While in the axial case the validity of the pseudospin symmetry was already demonstrated by the existence of nearly degenerate nucleon energy levels [44], the present study proves this validity directly for arbitrary nonaxiality even though the near degeneracy is no longer observable in the nucleon spectra.

The modified Nilsson Hamiltonian, employed in the study, has been constructed out of the same structural blocks as the pseudospin transformation itself. This is a natural generalization of a close connection between the two which is realized in the two known limiting cases. The structure of the spin-orbit and orbit-orbit terms combines the important features from the conventional triaxial Nilsson model (with the physical orbital momentum) [29, 40] as well as from the model with the deformed representation orbital momentum [39]. Namely, the orbital momentum-depending terms are optimally fitted to the conventional model while the Hamiltonian exactly preserves the number of quanta in the deformed oscillator representation. For this

reason, the current model displays an adequate behavior in the well-established regions of medium as well as very strong prolate deformations, and can be recommended for mean-field calculations in the broad deformation domain.

Since the pseudospin transformation of the model Hamiltonian cannot be accomplished in exact analytical form for an arbitrary deformation, a procedure of approximate transformation has been developed. This procedure generalizes the similar technique which was recently applied to the operators in the spherical representation and appeared to be rather accurate [5]. The approximate transforms, obtained in this chapter, happened to reproduce all of the known exact results in both the limiting cases. Incidentally, those exact results exist only for the operators of conserved quantities. Since the approximation procedure is based upon an operator-valued power expansion, the accuracy of the approximation could be increased by utilizing higher orders of the expansion. In this case, however, the transforms would acquire a complicated functional form with an explicit dependence on the shell number.

Outside the regions of weak and strong prolate deformations, the goodness of the approximation has been confirmed by both *a priori* and *a posteriori* tests. The *a priori* estimates utilize the analytical evaluation of some operator norms within the deformed oscillator shell that is conceptually close to the methods of statistical spectroscopy [47]. The *a posteriori* check is a numerical test based on a comparison of the spectra of the original Hamiltonian and its exact transform within the region of the unitarity of the pseudospin transformation. The tests demonstrate that the approximation procedure yields reliable results for all deformations and is most accurate for prolate-like shapes — which are observed in the majority of strongly deformed nuclei.

Apart from validating the approximation procedure, both the tests prove the physical equivalence between the spherical and deformed pseudospin representations at all experimentally attainable deformations. Thus, at the mean-field level the deformed "pseudo" Hamiltonian, which can be obtained by inserting the proper deformation dependence into the familiar pseudo Nilsson Hamiltonian of the spherical limit without any change in coefficients, occurs to be a very good approximation to the exactly transformed Hamiltonian (which cannot be written down in a closed analytical form). The equivalence condition is especially important for the many-particle pseudospin-adapted nuclear algebraic models like the pseudo SU(3) model and its pseudo symplectic extension which traditionally employ the spherical oscillator shell-model basis. This way of doing the calculation is therefore guaranteed to produce practically the same results as the calculation in the deformed pseudo shell-model basis provided the Hamiltonian is adequately transformed to the pseudospin representation in either case.

CHAPTER 5

SUMMARY AND CONCLUSIONS

Almost three decades have elapsed since pseudospin symmetry was discovered, and yet it is impossible to report that this symmetry is well understood and adequately described. Up to now there have been practically no attempts to look deeper into its essence — probably because it is often considered to be no more than a relabeling scheme and/or an accidental property of the oscillator shell model. It would be a pleasure to the author if this study creates a small gap in that attitude.

The most important result of the project is the identification of the many-particle momentum helicity transformation as the one responsible for passage to the pseudospin representation in real nuclei. This transformation has been selected among other candidates as the only one which provides a realization of the pseudospin relabeling rule and simultaneously satisfies the set of general symmetry requirements including unitarity, parity and time-reversal symmetry, rotational and translational invariance. Translational invariance is the crucial constraint in this list because it makes the choice unique.

A close relation between the helicity transformation and the pseudospin transformation of the oscillator shell model is displayed through comparative analysis of their action on the wavefunctions of the spherical oscillator. While values of pseudo-orbital and pseudospin momenta of a nucleon are the same in both cases, there are several properties of the transformations that make a difference. First, the helicity transformation is unitary in the entire space of states, while the unitarity of the pseudospin transformation is artificially confined to the normal parity sub-

space only. Moreover, the helicity-transformed functions are not the eigenfunctions of the oscillator; however, the dominant shell contribution is rather high (about 80%). Relative to the closest oscillator function, the helicity-transformed function is somewhat scaled in the bulk and has a tail behavior that decreases asymptotically as a negative power of the radial distance. Thus, the pseudospin transformation is a specific projection of its microscopic prototype on the oscillator basis. It utilizes special symmetries of the oscillator and by so doing gains some advantages — but, in contrast to the helicity transformation, outside the oscillator shell model its use is very restricted.

Both mean-field and many-particle estimates demonstrate that in the helicity-transformed representation the nucleons move in a finite-depth nonlocal potential with an effectively reduced spin-orbit strength. In accordance with the familiar estimates for the oscillator pseudospin transformation [9, 17], this reduced strength has different signs for the two nucleon types: it is attractive for protons and repulsive for neutrons. The many-particle consideration, based upon the Dirac–Brueckner density-dependent self-consistent parametrization of the nucleon-nucleon interaction in the medium, indicates a connection of the pseudospin symmetry to the boson-exchange nature of internucleonic forces. An attractive feature of the helicity transformation, which may be the initial unveiling of an exciting new project, is its coincidence with the chiral transformation in the region of asymptotic freedom.

Although the origin of pseudospin transformation has been traced down to the helicity transformation, these two operations are certainly not the same. The pseudospin transformation has unique features which violate the rules of the helicity transformation but produce significant benefits when used within the oscillator shell model: restoration of the dynamical SU(3) symmetry for heavy nuclei and a dra-

matic decrease in the effective number of interacting fermions. There exists one more advantage which was known earlier from some numerical results but now, after the explicit form of the transformation was found in the spherical limit, can be realized in all cases of practical interest. This advantage lies in finding the transforms of physical operators in analytical form and, moreover, in the existence of simple approximate transforms of some important operators.

The technique of analytical transforming the operators, which depend on the particle coordinates, momenta and spins, has been developed on the basis of special permutation relations among the rotational invariants comprising the transformation operator. The permutation relations, in turn, are the consequence of the symplectic osp(1|2) superalgebra obeyed by those rotational invariants. By using the above technique, the analytical transforms have been derived for several operators of physical interest, including some rotational scalars, the operators of spin and orbital momentum, and the quadrupole moment tensor. None of these transforms were previously known in analytical form although they constitute necessary input for the calculation of spectra and transitions within the framework of the pseudospin-based nuclear models.

In the cases of multipole operators the exact transforms cannot be written in a simple form. Nevertheless, in these cases approximate transforms are derivable that accurately extract dominant parts from the corresponding exact operators. To derive the approximate transforms, an efficient procedure has been proposed. It makes use of some operator-valued expansions whose precision increases with increasing oscillator shell number. (Recall that this is exactly the case with pseudospin symmetry which is observable in heavy nuclei and higher shells.) By applying this procedure, the approximate transforms for the spin, orbital momentum

and quadrupole tensor have been evaluated. The leading terms of the transforms turned out to be proportional to the original operators; and the proportionality coefficients, which previously were "empirically" known from numerical calculations within several shells, have been reproduced to 1% accuracy. If used in the scope of the pseudo SU(3) and pseudo symplectic algebraic approaches, the approximate transforms thus derived allow the many-particle Hamiltonian and transition operators within the pseudospin representation to be given in a simple analytic form and, therefore, lead to streamlining the calculational process.

The final part of the study is devoted to the problem of the deformation dependence of the pseudospin transformation. Prior to this study, the explicit form of the transformation was known only in the spherical and cylindrical limits. Castaños and collaborators, who found both expressions [13, 15], underscored the difference between the two but gave no suggestion regarding a physical reason behind this difference or a way to reconcile what seemed to be contradictory results. The solution, proposed in this project, essentially uses the relation between the momentum helicity and pseudospin transformations [6]: the latter transformation is a specific adaptation of the former for the needs of the oscillator basis. Indeed, the many-particle helicity transformation is the universal microscopic operation, and it carries no explicit deformation dependence. However, in the mean-field approximation, information about the equilibrium shape of the nucleus is required from the onset and this dictates the choice of the basis functions. (In particular, the oscillator shell-model consideration of a deformed nucleus is most conveniently obtainable in the appropriately deformed oscillator basis.) The transformation, constructed this way, is fit to both the limiting cases and resolves the above dilemma.

However, the construction of the transformation is just one side of the problem. To be sure that the proposed approach is adequate, one should test directly whether the pseudospin representation descriptions of the same nucleus at different deformations are physically equivalent. This can be achieved by choosing a reasonable Hamiltonian and checking whether the deformation dependence of the terms, corresponding to physically similar interactions, is similar in the normal and pseudo representations. The positive result of this study suggests the adequacy of the proposed form of the deformed pseudospin transformation and indirectly the basic idea on the origin of the pseudospin symmetry.

It is noteworthy that to reach a positive outcome required solving two auxiliary problems of independent theoretical interest. One is the construction of the appropriate Hamiltonian; the other is its transformation to the pseudospin representation.

The problem with the Hamiltonian is that it is well established in the region of low and medium deformation where most of the experimental data are found (it is the familiar Nilsson scheme [29, 40]), but the structure of its terms at strong deformation is still under question (except for the asymptotic scheme for very high prolate deformation). To arrive at the solution, use has been made of the fact that in both the limiting cases the Hamiltonian is constructed out of the same structural blocks as the pseudospin transformation itself, and the construction algorithm is very similar. The desired Hamiltonian has been designed as a generalization of this algorithm so that both limits were naturally incorporated. A substantial feature of the Hamiltonian is that while its eigenstates cannot be labeled by the physical angular momentum in presence of deformation, they are as close to the angular momentum eigenstates as possible under all the constraints of the model.

The analytical expressions for pseudospin transforms cannot be derived in closed form, even for the model Hamiltonian — in contrast to both of the known limiting cases. However, they can be derived approximately. It is interesting that the procedure for performing this derivation has been developed as a generalization of a similar procedure employed for transforming the physical operators in the spherical limiting case. The operator-valued expansion, which is basic to the procedure, has been naturally truncated at a degree, which yields the transforms on the same level of complexity as the original operators themselves. For this reason, the transformed Hamiltonian consists for all practical purposes of the same operators as in the original one which, as mentioned above, confirms the adequacy of the pseudospin representation at arbitrary deformations. The approximately transformed Hamiltonian has been found to strictly coincide with the exactly known limiting cases, and the strength of the pseudo spin-orbit interaction has been evenly reduced compared to the original strength at all deformations. Both analytical and numerical performance evaluations of the approximation procedure show very satisfactory results in the deformation domain expanding to and even outside of the hyperdeformation area. In the context of the above paragraphs it is tantamount for proving the goodness of the pseudospin symmetry at any experimentally attainable deformation.

This project reports on progress towards understanding issues of fundamental interest in nuclear physics like the origin of pseudospin, its relation with the helicity transformation and symmetries of nucleon interaction, and validity of the pseudospin dynamical symmetry. The advance would not have been possible without adequate tools — and on the way to those issues an appropriate formalism has been proposed and developed that can prove useful elsewhere. Although a specific study is expected to reach its final conclusion, the research process itself is never complete. There are

always more unsolved problems at different levels of consideration than satisfying answers: Is it possible to incorporate the helicity transformation into the structure of realistic nuclear models and gain the advantages similar to the benefits of the pseudospin? Once the helicity transformation is close to the chiral transformation in the high-energy limit, could it also be useful for the quark models? Is there a convenient way to perform many-particle, shell-model calculations in the deformed pseudospin representation? What is the proper Hamiltonian for this case within the pseudo SU(3) model?.. New riddles emerge every day, and those that merit answers will eventually gain them.

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APPENDIX A

HELICITY TRANSFORMATION FOR OSCILLATOR WAVEFUNCTIONS

The action of the helicity transformation on a function $\phi(\mathbf{r})$ of the coordinate and spin variables is conveniently representable by means of an auxiliary Fourier transformation:

$$\mathcal{H}\phi(\mathbf{r},\sigma) = \mathcal{F}^{-1}\left(\mathbf{e_k}\cdot\sigma\,\mathcal{F}\phi(\mathbf{r},\sigma)\right), \tag{A.1}$$

where

$$\mathcal{F}\phi(\mathbf{r},\sigma) = \frac{1}{(2\pi)^{3/2}}\int d\mathbf{r} e^{i\mathbf{k}\cdot\mathbf{r}}\phi(\mathbf{r},\sigma), \tag{A.2}$$

$$\mathcal{F}^{-1}\phi'(\mathbf{k},\sigma) = \frac{1}{(2\pi)^{3/2}}\int d\mathbf{r} e^{-i\mathbf{k}\cdot\mathbf{r}}\phi'(\mathbf{k},\sigma) \tag{A.3}$$

denote the direct and inverse Fourier transforms, respectively, and $\mathbf{e_k} = \mathbf{k}/k$ is the unit vector in the $\mathbf{k}$ direction.

By utilizing the spherical wave expansion of a vector plane wave

$$e^{i\mathbf{k}\cdot\mathbf{r}} = 4\pi\sum_{l\geq 0} i^l j_l(kr)\,\mathbf{Y}_l(\mathbf{e_r})\cdot\mathbf{Y}_l(\mathbf{e_k}), \tag{A.4}$$

the normalization condition for spherical harmonics, and the formula

$$\mathbf{e_k}\cdot\sigma\, i^l(\mathbf{Y}_l(\mathbf{e_k})\otimes\chi)_{jj_z} = i^{\bar{l}}(\mathbf{Y}_{\bar{l}}(\mathbf{e_k})\otimes\chi)_{jj_z} \tag{A.5}$$

(which was obtained in Ref. [9] for the coordinate space), it is straightforward to derive the transform (2.5) for the oscillator function (2.4) in the spherical representation. The transformed radial function $\mathcal{R}_{\tilde{n}\tilde{l}q}(r)$ is then calculable with the help of the Hankel integral transformation:

$$\mathcal{R}_{\tilde{n}\tilde{l}q}(r) = \sqrt{\frac{2}{\pi}}\int_0^\infty dk k^2 j_{\tilde{l}}(kr) P_{nl}(k), \tag{A.6}$$

$$P_{nl}(k) = \sqrt{\frac{2}{\pi}}\int_0^\infty dr r^2 j_l(kr) R_{nl}(r), \tag{A.7}$$

where the change $l \to \tilde{l}$ in the index of the spherical Bessel function occurs as a result of the spin-angular transformation (A.5). (The definition of $\bar{l}$ and $\tilde{n}$ is given in Eq.(2.6); it implies that the number of radial nodes $\nu = (n-l)/2$ is conserved under the direct Fourier transformation.)

It is convenient to introduce a family of dimensionless oscillator radial functions

$$U_{\nu l}(x) = \sqrt{\frac{\nu!}{2\pi\Gamma(\nu+l+3/2)}}\, x^l L_\nu^{l+1/2}(x^2) e^{-x^2/2}. \tag{A.8}$$

Then, from the symmetry of the spherical oscillator Hamiltonian with respect to the Fourier transform it follows that

$$R_{nl}(r) = \frac{1}{r_0^{3/2}} U_{(n-l)/2,l}\left(\frac{r}{r_0}\right), \tag{A.9}$$

$$P_{nl}(k) = (-1)^{(n-l)/2} r_0^{3/2} U_{(n-l)/2,l}(kr_0), \tag{A.10}$$

where $r_0 = \sqrt{\hbar/(m\omega)}$ is the oscillator radius. The inverse transform can be performed by using the explicit form of the Laguerre polynomials,

$$L_\nu^{l+1/2}(x^2) = \sum_{m=0}^{\nu} \frac{(-1)^m}{m!} \binom{\nu + l + 1/2}{\nu - m} x^{2m}, \tag{A.11}$$

and the integral

$$\begin{aligned} &\int_0^\infty dt\, t^{\kappa-1} j_l(xt) e^{-t^2/2} \\ &= 2^{(\kappa-l-2)/2} \frac{\Gamma((l+\kappa)/2)}{\Gamma(l+3/2)} x^{l+1/2} {}_1F_1\left(\frac{l-\kappa+3}{2}, l+\frac{3}{2}, \frac{x^2}{2}\right) e^{-x^2/2} \end{aligned} \tag{A.12}$$

(which is a particular case of Eq. (11.4.28) in Ref. [31]), and results in the formula

$$\mathcal{R}_{\tilde{n}\tilde{l}q}(r) = \frac{1}{r_0^{3/2}} \mathcal{U}_{(\tilde{n}-\tilde{l})/2,\tilde{l},q}\left(\frac{r}{r_0}\right). \tag{A.13}$$

Here, by definition,

$$\begin{aligned} \mathcal{U}_{\nu\ell q}(x) = (-1)^\nu 2^{q/2} \sqrt{\frac{\nu!\,\Gamma(\nu+\ell+q+3/2)}{2\pi}}\, \Gamma(\ell+3/2)\, x^\ell e^{-x^2/2} \\ \times \sum_{m=0}^{\nu} \frac{(-2)^m\, \Gamma(\ell+m+(q+3)/2)}{m!(\nu-m)!\Gamma(\ell+m+q+3/2)} {}_1F_1\left(-m-\frac{q}{2}, \ell+\frac{3}{2}, \frac{x^2}{2}\right) \end{aligned} \tag{A.14}$$

for any nonnegative integer ν, ℓ, and $q = -1, 0$ or 1. (Only the values $q = 1$ or -1 matter for the transformed functions; $q = 0$ makes sense just for an error check since $\mathcal{U}_{\nu\ell 0}(x) \equiv U_{\nu\ell}(x)$.)

The example of behavior of the transformed radial functions compared to the closest oscillator radial function is given in Fig. 2.1 for $\nu = 1$, $\ell = 3$. As one can see, the bulk behavior of the three functions is very similar, while $U_{\nu\ell}$ is the dominant component in the oscillator function expansion of both $\mathcal{U}_{\nu\ell+}$ and $\mathcal{U}_{\nu\ell-}$.

Moreover, it will be shown below that both the latter functions in the bulk are with good accuracy obtainable from the former by a mere scaling transformation and subsequent normalization; and this scaling rule is valid for more realistic mean fields as well.

The expansion coefficients $c_{\nu\ell q;\nu'}$, occurring in the series

$$\mathcal{U}_{\nu\ell q}(x) = \sum_{q\geq 0} c_{\nu\ell q;\nu'} U_{\nu'\ell}(x), \tag{A.15}$$

are directly calculable either in coordinate or in momentum space. The latter way is definitely simpler since in this case the orbital momentum label is the only parameter to be changed. Then the calculation is reducible to expanding the $x^{q/2}L_{\nu}^{\ell+q+1/2}$ function in terms of $L_{\nu'}^{\ell+1/2}$ polynomials, and results in

$$c_{\nu\ell q;\nu'} = \sqrt{\frac{\nu!\,\nu'!}{\Gamma(\nu+\ell+q+3/2)\,\Gamma(\nu'+\ell+3/2)}}\left(\ell+\frac{q+1}{2}\right)! \tag{A.16}$$
$$\times \sum_{k=0}^{\min(\nu,\nu')} \binom{\ell+(q+3)/2+k-1}{k} \binom{q/2}{q/2-\nu'+k} \binom{-q/2}{-q/2-\nu+k}.$$

The diagram of these coefficients versus the shell number is given in Fig. 2.2 for the same case as in Fig. 2.1. The total contribution from higher shells is fairly small for both values of q; however, for $q=-1$ the expansion converges more slowly because of the larger tail of the corresponding radial function.

It is noteworthy that there exists another example of a wavefunction whose helicity transform is easily calculable analytically. Indeed, by using Eq. (A.5), the Hankel transformation procedure and the normalization integral

$$\int_0^\infty dr\; r^2 j_l(kr) j_l(\kappa r) = \frac{\pi}{2k\kappa}\delta(k-\kappa), \tag{A.17}$$

one can prove the following formula:

$$\mathcal{H}\, i^l j_l(\kappa r)(\mathbf{Y}_l(\mathbf{e}_r)\otimes\chi)_{jj_z} = i^{\bar{l}} j_{\bar{l}}(\kappa r)(\mathbf{Y}_{\bar{l}}(\mathbf{e}_r)\otimes\chi)_{jj_z}. \tag{A.18}$$

The importance of this equation for realistic mean-field nuclear models is underscored by the fact that within the bulk the leading term of a bound-state wavefunction with the angular momentum j, the orbital momentum l and the energy $\epsilon = -\hbar^2\kappa^2/(2m)$ is proportional to $j_l(\kappa r)(\mathbf{Y}_l(\mathbf{e}_r)\otimes\chi)_{jj_z}$. Therefore, after the helicity transformation the binding energy is conserved, the orbital momentum changes from l to $\bar{l}$ according to the rule (2.6), and the central potential remains rather flat in the nuclear bulk. This model-independent result reemphasizes the understanding of the helicity transformation as a microscopic precursor of the pseudospin transformation.

For the spherical oscillator eigenstates, $\kappa = \sqrt{2n+3}/r_0$, and the radial function $R_{nl}(r)$ behaves like $j_l(\sqrt{2n+3}r/r_0)$ within the bulk. According to the rule (A.18), the transformed function $\mathcal{R}_{\tilde{n}\bar{l}q}(r)$ is then proportional to $j_{\bar{l}}(\sqrt{2n+3}r/r_0)$ which in turn is equivalent to the scaling relation,

$$\mathcal{R}_{\tilde{n}\bar{l}q}(r) \propto R_{\tilde{n}\bar{l}}\left(\sqrt{\frac{2n+3}{2\tilde{n}+3}}r\right), \tag{A.19}$$

where $n = \tilde{n} + q$ in correspondence with (2.6). This scaling rule is mentioned above in conjunction with Fig. 2.1.

APPENDIX B

PERMUTATION RELATION FOR SPIN

The purpose of this section is to derive the permutation relation (3.14) for the spin operator and an arbitrary analytic function of $\mathbf{l}\cdot\boldsymbol{\sigma}$.

Note that the anticommutation rule

$$\boldsymbol{\sigma}(\mathbf{l}\cdot\boldsymbol{\sigma}) + (\mathbf{l}\cdot\boldsymbol{\sigma})\boldsymbol{\sigma} = 2\mathbf{l}$$

can be rewritten in the form

$$\boldsymbol{\sigma}(\mathbf{l}\cdot\boldsymbol{\sigma}) = -(\mathbf{l}\cdot\boldsymbol{\sigma} + 1)\,\boldsymbol{\sigma} + 2\mathbf{j}. \tag{B.1}$$

Since the $\mathbf{l}\cdot\boldsymbol{\sigma}$ operator commutes with $\mathbf{j}$, this equation can be used recursively for permuting $\boldsymbol{\sigma}$ with a power function of $\mathbf{l}\cdot\boldsymbol{\sigma}$. Assume that a general solution for this recursion goes as follows:

$$\boldsymbol{\sigma}(\mathbf{l}\cdot\boldsymbol{\sigma})^k = (-\mathbf{l}\cdot\boldsymbol{\sigma} - 1)^k\boldsymbol{\sigma} + 2\xi_k(\mathbf{l}\cdot\boldsymbol{\sigma})\mathbf{j} \tag{B.2}$$

where $\xi_k(x)$ is an unknown function and $\xi_1(x) = 1$. Multiply Eq. (B.2) by $\mathbf{l}\cdot\boldsymbol{\sigma}$ from the right and use Eq. (B.1) to arrive at the relation,

$$\xi_{k+1}(x) = x\xi_k(x) + (-x - 1)^k,,$$

which has a solution

$$\xi_k(x) = \frac{x^k - (-x-1)^k}{2x+1}.$$

The latter formula provides the missing element in the permutation relation (B.2) which in turn yields Eq. (3.14) as a consequence of the linearity of the original expression.

It is of some interest to note that the operator-valued function

$$\frac{f(\mathrm{l}\cdot\sigma) - f(-\mathrm{l}\cdot\sigma - 1)}{\mathrm{l}\cdot\sigma + 1/2},$$

which occurs as a coefficient of j in Eq. (3.14), is in fact spin independent. To determine this result, observe that the function is symmetric under the substitution $\mathrm{l}\cdot\sigma \to -\mathrm{l}\cdot\sigma - 1$. A simple analysis shows that it actually depends only on the $\mathrm{l}\cdot\sigma(\mathrm{l}\cdot\sigma + 1)$ combination which is just l^2 (cf. Eq. (3.20)).

APPENDIX C

IDENTITIES FOR DEFORMATION MATRIX

The dimensionality of the physical space places very strict constraints on the number of linearly independent functions of the frequencies ϵ_z, ϵ_x and ϵ_y. This is also a principal behind the choice of the independent operator set for which the approximate pseudospin transforms are derived in Sec. 4.

To explicitly determine the dimensionality constraints, it is convenient to introduce the deformation matrix

$$\epsilon = \mathrm{diag}(\epsilon_x, \epsilon_y, \epsilon_z). \tag{C.1}$$

Generally speaking, ϵ is a second rank symmetric tensor. Since the discussion refers to the principal frame under the condition of volume conservation, it is assumed to be both diagonal and unimodular ($\epsilon_z\epsilon_x\epsilon_y = 1$).

The quintessential of the dimensionality restrictions for matrices is expressed in the Cayley–Hamilton theorem: A square matrix obeys its characteristic equation (see, for example, Ref. [35]). For the deformation matrix the characteristic determinant is

$$\det(\rho\mathbf{I} - \epsilon) = \rho^3 - m_1\rho^2 + m_{-1}\rho - 1, \tag{C.2}$$

where $\mathbf{I}$ denotes the unit matrix. Consequently, the matrix equation is written in the form

$$\epsilon^3 - m_1\epsilon^2 + m_{-1}\epsilon - \mathbf{I} = 0, \tag{C.3}$$

which makes explicit use of the unimodularity and the definition (4.44) of the invariants of ϵ. By induction, it follows from this equation that an arbitrary analytical function $f(\epsilon)$ is expressible as a linear combination of three basis matrices with the coefficients which are invariants of ϵ. The choice of the three basis matrices is by no means unique; one of the most convenient options is ϵ, I and ϵ^{-1}.

Furthermore, any operator of the form,

$$\mathrm{Tr}(f(\epsilon)\hat{\mathbf{x}}), = \sum_s f(\epsilon_s)\hat{x}_s$$

can then be rewritten as a linear combination of only three independent operators. In particular, the operator $\sum_s f(\epsilon_s)\mathsf{b}_s\sigma_s$ can be rewritten as a linear function of the operators d, $\sum_s \mathsf{b}_s\sigma_s$ and d' (see Eqs. (4.33) and (4.76)). This in turn leads to the proof of the fact that the same operator, $\sum_s f(\epsilon_s)\mathsf{b}_s\sigma_s$, is expressible in terms of d and d^+ only. Indeed, the iterative use of the commutation relation

$$[\sum_s \epsilon_s^k \mathsf{b}_s\sigma_s, h_{osc}] = \sum_s \epsilon_s^{k+1}\mathsf{b}_s\sigma_s \tag{C.4}$$

allows the construction of the operator $\sum_s \epsilon_s^k \mathsf{b}_s\sigma_s$ for virtually any k starting from the d operator. By combining this relation with the Hadamard operator identity,

$$e^{-\hat{x}}\hat{y}\,e^{\hat{x}} = \sum_{k\geq 0}\frac{1}{k!}\underbrace{[\ldots[[[\hat{y},\hat{x}],\hat{x}],\hat{x}],\ldots,\hat{x}]}_{k\text{-fold commutator}}, \tag{C.5}$$

the following formula can be obtained:

$$e^{\rho h_{osc}} d e^{-\rho h_{osc}} = \sum_s \sqrt{\epsilon_s} e^{-\rho\epsilon_s}\mathsf{b}_s\sigma_s. \tag{C.6}$$

The integration of the latter operator identity over the parameter ρ yields the relations

$$\int_{\rho=0}^{\infty} d\rho\, e^{\rho h_{osc}} d e^{-\rho h_{osc}} = d', \tag{C.7}$$

$$\int_{\rho=0}^{\infty} \frac{d\rho}{\sqrt{\pi\rho}}\, e^{\rho h_{osc}} d e^{-\rho h_{osc}} = \sum_s \mathsf{b}_s \sigma_s, \tag{C.8}$$

which prove the desired result since h_{osc} is also expressible via d and d^+ (see Eq. (4.41)).

Finally, several identities are listed below for the functions and invariants of the deformation matrix. They follow from Eq. (C.3) and the unimodularity condition and are utilized to derive various equations in this paper (predominantly in Sec. 4):

$$\epsilon^2 = 3m_1\epsilon - 3m_{-1}\mathbf{I} + \epsilon^{-1}, \tag{C.9}$$

$$\epsilon = m_{1/2}\epsilon^{1/2} - m_{-1/2}\mathbf{I} + \epsilon^{-1/2}, \tag{C.10}$$

$$\epsilon^{1/2}(\epsilon - m_1\mathbf{I}) = m_{-1/2}\epsilon^{1/2} - (m_{1/2}m_{-1/2} - 1)\mathbf{I} + m_{1/2}\epsilon^{-1/2}, \tag{C.11}$$

$$m_k{}^2 = m_{2k} + 2m_{-k}, \tag{C.12}$$

$$4m_3 = (m_2 - m_{-1})m_1 + 3. \tag{C.13}$$

VITA

Andrey Blokhin was born on January 27, 1965 in Kiev, Ukraine. He studied physics at Kiev National University and graduated in 1986 after defending a thesis. He continued his research at the Department of Nuclear Structure of the Bogolyubov Institute for Theoretical Physics under the supervision of Professor G.F. Filippov, D.Sc., and simultaneously studied at the Graduate School of the Ukrainian Academy of Sciences. He defended a dissertation in 1990 at the Council for Dissertation Defenses of Kiev National University and received the confirmation of the Candidate of Science degree from the Highest Attestation Committee of the Union of Soviet Socialist Republics. He entered the doctoral program of Louisiana State University in 1992 and worked at the Nuclear Theory group of the Department of Physics and Astronomy under the supervision of Professor J.P. Draayer. His Doctor of Philosophy degree will be awarded in August of 1996.

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DOCTORAL EXAMINATION AND DISSERTATION REPORT

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